Voting Power in the EU Council of Ministers and Fair Decision-Making (very preliminary draft)

Michel Le Breton Vera Zaporozhets

May 2010

Abstract

We analyze and evaluate the different decision rules describing the Council of Ministers of the EU starting from 1958 up to date. All the existing studies use the Banzhaf and the Shapley-Shubik indices. We argue that the nucleolus can also be considered as an appropriate power measure. We develop an algorithm to calculate the nucleolus and compare the results of our calculations with the conventional measures. In the second part, we analyze the power of the European citizens as measured by the nucleolus under the egalitarian model proposed by Felsenthal and Machover (1998), and characterize the first best situation. Based on these results we propose a methodology for the design of the optimal (fair) decision rules.

1 Introduction

Democratic decision-making, in local, national or supra-national bodies, is based on voting. Political scientists and economists alike have long noted that it is far from obvious to evaluate the voting power of different individuals or groups, e.g. parliamentary coalitions, in decision-making bodies. They noticed that the voting power need not be proportional to the relative number of votes an individual or a group is entitled to. For example, Luxembourg was powerless in the Council of Ministers of the EU between 1958 and 1973. It held one vote, whereas a qualified majority of votes was defined to be 12 out of 17. Since other member states held an even number of votes, Luxembourg formally was never able to make any difference in the voting process. The recent enlargement of the European Union caused a lively debate on the adequate tools for measuring decision power in real-life institutions and had strong implications for the balance of the power among member states.

During the last decade scholars have continued to contribute to the theoretical and empirical research on power indices[1]. One of the important applied questions addressed in this literature is whether the national representation in the European Union is fair or not. It has often been claimed that the current allocation of votes among EU states is not fair. In particular, it is often asserted that, in the European decision-making process, the large countries are under-represented while the reverse holds for the small ones. In this paper we address this question by performing the evaluation of the power distribution among the member states in the EU Council of Ministers starting from 1958 up to date using the nucleolus. We conclude that in most of the cases, the above critique is justified, and therefore we propose a new methodology for the design of the optimal (fair) decision rules. In particular, we show that in the Council of Ministers in 1958, Germany got too little weight as compared to France and Italy, and that, surprisingly, the choice to make Luxembourg a dummy was optimal in our context. In what follows, we explain why the nucleolus is an appealing power measure for this analysis.

As noted by Napel and Widgren (2004) "Scientists who study power in political and economic institutions seem divided into two disjoint methodological camps. The first one uses non cooperative game theory to analyze the impact of explicit decision making procedures and given preferences over a well-defined, usually Euclidean policy space. The second one stands in the tradition of cooperative game theory with more abstractly defined voting bodies: the considered agents have no particular preferences and form winning coalitions which implement unspecified policies. Individual chances of being part of and influencing a winning coalition are then measured by a power index....Proponents of either approach have recently intensified their debate which was sparked by the critique by Garrett and Tsebelis (1999, 2001).... *Several authors have concluded that it is time to develop a unified framework for measuring decision power*. On the one hand, such framework should allow for predictions and ex post analysis of decisions based on knowledge of procedures and preferences. On the other hand, it must be open to ex ante and even completely a priori analysis of power when detailed information may either not be available or should be ignored for normative reasons".

[1] See for instance, Algaba et al. (2007), Barr and Passarelli (2009), Bilbao et al. (2002), Felsenthal and Machover (2001, 2004), Laruelle and Widgren (1998) and Leech (2002).

Some authors including Steunenberg, Schmidtchen and Koboldt (1999), Maaser and Napel (2007), Napel and Widgren (2004) have proposed models of public decision making where, to some extent, the two points of view are reconciled. Steunenberg, Schmidtchen and Koboldt (1999) propose a general framework where the policy space is a multidimensional space and preferences are defined by the Euclidean distance to an ideal point. The power of a player with respect to an arbitrary outcome function (i.e., a function mapping a profile of ideal points into a policy) is defined as the expected payoff of this player and the expected payoff of a random player. They apply their theory to the case where the policy space is one-dimensional and where the game form is intended to model the EU decision-making process. Maaser and Napel (2007) also consider a one-dimensional policy space and model a two-tier representative system where each citizen in each constituency has a single peaked symmetric preferences. They assume that the representative of each constituency is the median voter of the constituency and that the decision taken at the top tier is the position of the pivotal representative. Using Monte-Carlo simulations, they investigate several artificial constituency configurations as well as the EU and the US electoral college. Precisely, given a random device to select the ideal points, they look for the allocation of voting weights for which each voter in each constituency have an equal chance to determine the policy implemented by the top tier, and show that the Penrose square root principle comes close to ensuring equal representation. Napel and Widgren (2004) consider the situation where the status quo is matched against a proposal but the decision to challenge the status quo as well as the nature of the proposal is not exogenous like in traditional models of power measurement; instead the proposal is under the control of an agenda setter. They sketch a theory of power measurement (the players are the voters and the agenda-setter) for this specific setting and under the extra assumption of unidimensionality: where (ex ante) power is defined as expected marginal influence[2].

Napel and Widgren (2004) assert that "So far, we have only considered ideal points in one-dimensional policy spaces. These are analytically convenient. Both the derivation of ex post power and formation of expectations are more complicated for higher-dimensional spaces. *However, there is no obstacle, in principle*". In this paper, we aim to contribute to the reconciliation between the two approaches. We certainly agree with the postulate that game forms have to be taken into account by political analysis but we do not want the power analysis to be extremely sensitive to the details of the game form used to describe the non-cooperative decision process, i.e. we would like to derive some *robust* power measure. To do so, we consider a specific, but extremely important, multidimensional policy space, namely *distributive politics.* Precisely, we are interested in multidimensional policy issues which can be represented as vectors in the simplex of some Euclidean space. This setting arises naturally when the issue under scrutiny is the allocation of a fixed budget (surplus, cost, gains from cooperation or coordination,...) across the members of an organization. More generally, under the assumptions of transferable utility (i.e., quasi-linearity with respect to some common numeraire) and efficiency in public decision making, the simplex structure

[2]Several authors including Napel and Widgren (2006, 2009), Passarelli and Barr (2007 and Tsebelis (1994) have analysed non cooperative game forms describing the interaction between the decision bodies (among which the council of ministers) involved in the EU decision making process.

appears as the efficient frontier of any bounded and convex subset of policies like those considered in the spatial model of politics. The unit of measurement will be interpreted below as being money but alternative units, like for instance ministry portfolios or (local) public expenditures, can be considered. The key assumption that we make on preferences is that members of the organization only care about their share. This means that the ideal points are the vertices of the simplex and that there is no room here for a difference between ex ante and ex post power measurement from the perspective of preferences.

Several alternative forms can be considered to describe the public decision making process. Following Montero (2006) and Snyder et al. (2005)), we could consider for instance a *legislative bargaining game a la Baron and Ferejohn* where players act strategically as proposers and (or) voters. It has been shown under some conditions on the vector of probabilities of being selected as a proposer and on the structure of the simple game describing the distribution of voting power in the organization, that the nucleolus of the simple game is the unique vector of expected equilibrium payoffs[3]. Another game form for which the nucleolus also appears as the vector of equilibrium payoff is the celebrated *sequential lobbying model* pioneered by Groseclose and Snyder (1996) and further explored by Banks (2000), Diermeier and Myerson (1999), Le Breton and Zaporozhets (2010) and Le Breton, Sudhölter and Zaporozhets (2010) among others. In this model, two competing lobbies buy the votes of the (some of the) members of a legislature in order to get these people to vote for their most preferred alternative. Young (1978 a, b) had already developed a quite similar model in a series of illuminating papers. Young (1978 a,b), Le Breton and Zaporozhets (2010) and Le Breton, Sudhölter and Zaporozhets (2010) have independently demonstrated that if at equilibrium lobbying takes place, then the nucleolus is a vector of equilibrium payoffs and, often, the unique vector of equilibrium payoffs. In both models, the ex ante approach is well defined. In the bargaining model, it is attached to the vector of probabilities of being selected to act as a proposer. In the lobbying model, as suggested in Diermeier and Myerson (1999), randomness results from the fact that the willingness to pay of each lobby is the realization of a random variable and that lobbying takes place iff the ratio of the two realizations is larger than some threshold called the *hurdle factor*.

These arguments provide grounds for our choice of the nucleolus as a contender to the traditional measures. In contrast to the Banzhaf and the Shapley-Shubik power indices which are very well defined in the context of a binary ideological setting[4], the above line of reasoning

[3] In this problem where an alternative is a division of a pie, the nucleolus predicts a larger share for those who have a larger weight in the voting process. This raises a number of emprical questions, in particular, whether reapportionment in a legislature affects policy outcomes? Since the United States Supreme court established the principle of "one person-one vote" in the 1960s, a number of american legal scholars and political scientists have examined this question (see e.g. Ansolabehere, Gerber and Snyder (2002)). Horiuchi and Saito (2003) examine the same question from a comparative perspective by focusing on reapportionnement associated with the electoral reform implemented in Japan in 1994. It was like a natural experiment. The reform has reduced the overall level of reapportionnement in a very short span of time and they were therefore in position to examine the effects of reapportionnement on policy outcomes while holding other social, economic, and demographic factors almost at a constant level. Both empirical studies conclude that the influence is significant.

[4] We refer to the monographs of Felsenthal and Machover (2003) and Laruelle and Valenciano (2008) for a description of the state of the art and for a rigorous and comprenhensive treatment of the binary setting.

shows that in the context of the distributive politics the nucleolus can be considered as an appropriate power measure. This point of view has been advocated a long ago with force and talent by Young[5] (1978 c) and more recently by Montero (2005). In addition to the non-cooperative foundations which have just been mentioned, these papers argue convincingly that the nucleolus is a very consistent power index. Our paper aim to contribute to the diffusion of the idea that the nucleolus is indeed a power index that should be considered in applied positive and normative analysis of organizations described as weighted majority game.

In the first part of our paper, we evaluate different decision rules for the Council of Ministers of the EU starting from 1958 up to date. We develop an algorithm to compute the nucleolus and apply it to analyze the distribution of voting power in the Council of Ministers of the European Union. We compare the results of our calculations with the predictions provided by the Banzhaf and the Shapley-Shubik as well as another index obtained from the non-cooperative bargaining game due to Baron and Ferejohn (see Montero (2007)). We are interested both in the power of a country to approve as well as its power to block a decision. The Banzhaf and the Shapley-Shubik indices give the same answer in both situations. The two new measures may assign different values to the power to approve and to the power to block a proposal by a country.

In the second part, we move to a normative analysis, i.e. to the determination of the weights that should be assigned to the members of the EU council of ministers in order to achieve a certain social objective[6]. Hereafter, we will refer to this weights as being the *optimal weights.* In the classical binary setting, this approach has been pursued by many authors. For instance, Machover and Felsenthal (1998) ask how the optimal weights look like in the Banzhaf setting when the objective is to minimize the majority deficit or equivalently to equalize the power of the citizens. They show that the optimal weights are proportional to the square root of the respective constituency's population sizes, the celebrated Penrose's rule (1946). In Barbera and Jackson (2006), the optimal weights result from a the maximization of an utilitarian objective. They found that the optimal weights will depend upon the details of the probability process selecting the profile of utilities. This utilitarian model has been explored further by Beisbart, Bovens and Hartmann (2005) and Beisbart and Hartmann (2010); these papers reinforce the recurrence of the Penrose recommendation as an optimal solution. In this paper, we follow the egalitarian approach with the nucleolus being the measure of power of the countries in the EU Council of Ministers: in our setting the role of the Council of Ministers is to distribute some surplus across the countries. The country amount is then divided equally among their citizens (we do not introduce any bias). If this surplus is interpreted as the gains from the EU, we would like this surplus to be shared equally among European citizens[7]. It follows from our result that this egalitarian

[5] In Young (1978 c) a new and different approach to power measurement is developped.

[6] The selection of national voting weights in the Council of Ministers of the European Union and its implied influence on the EU legislation have received a great deal of attention from academics, politicians and the general public and has generated a lot of controversies.

[7] The principle of "one person, one vote" is generally taken to be a corner stone of democracy. In this distributive setting, the principle is as simple as "one person, one euro".

goal will be met perfectly if and only if the nucleolus for the representatives equals to the population shares. It is not clear, however, how this principle ought to be operationalized in practice either in terms of apportioning an integer number of seats for given non-integer ideal shares or in determining what are the ideal shares. Although it seems straightforward to allocate weights proportional to population sizes, this ignores the combinatorial properties of weighted voting, which often imply spark discrepancies between voting weight and actual voting power as illustrated in the beginning of the introduction. We are confronted to a truly *combinatorial second best optimization problem*[8]. Second best, because we will never reach the perfection and we need therefore to evaluate the social loss associated to any deviation from perfect equality. Combinatorial, because we have only a finite number of possibilities. In that respect the terminology "optimal weights" can be misleading as what really matters is the simple game induced by the weights. If there were only three countries, the notion of weights is almost meaningless. In addition to that, let us also point out that there is no reason to infer that the second best optimal simple game will be a weighted majority game[9]. The combinatorial problem is difficult[10]. We introduce a methodology, based on the specific criterion of *variance minimization*[11], for the design of the voting rule. Implementing the method is far from being easy. We illustrate its application when the number of EU members was very small.

The rest of the paper is organized as follows. In the subsequent subsection we provide a review of the closely related literature. Section 2 describes the first five configurations of the Council of Ministers between 1958 and 1995 which operated under the weighted voting rules. We provide the values for the nucleolus and the expected payoffs from the corresponding bargaining game both for the approval and the block situations. The expected payoffs, in fact, are given only up to 1986 due to the computational complexity. We compare these values with the more traditional power measures as the Banzhaf and the Shapley-Shubik indices. In section 3, we describe the qualified voting rules for 15 and 27 members as prescribed by the Treaty of Nice, and compare the nucleolus with the values for the Banzhaf and the Shapley-Shubik indices. Section 5 is devoted to the design of the optimal (fair) decision rules. An appendix is dedicated to an overview of the notions from cooperative game theory which are used in this paper, as well as some results on the combinatorics of simple games with a special attention to the issue of representation by weights.

[8] Barbera and Jackson (2006) refer to this as equivalent voting rules.

[9] In the utilitarian framework, as demonstrated by Barbera and Jackson, the optimal voting rule is almost a weighted voting rule and is a voting rule under some specific assumptions.

[10] This explains why many practionners select a parametrized family of weight functions (for instance, the population of the country to the power α) and calculate the values of the power indices resulting from each feasible choice of the parameter(s). It is not entirely clear to us why this procedure guarantees that the optimal second best simple game can be determined through such exploration.

[11] Variance minimization has been adopted by many authors. Of course, many other inequality indices like for instance, the Gini index and the Kolm-Atkinson's indices could be used instead. In this paper we have not explored the sensivity of the conclusions to the choice of a particular index.

2 Two "New" Power Indices

A measure of power is a map ξ from the set of simple games $(N, \mathcal{W})$ to the set of n-tuples of real numbers. The value $\xi_i = \xi_i (N, \mathcal{W})$ is the power of player i in the game $(N, \mathcal{W})$, and it satisfies $0 \leq \xi_i \leq 1$. The most famous power measures used in the literature are the Banzhaf (BZ) and the Shapley-Shubik (SS) indices[12]. In this paper, we introduce two new measures of power which are not derived from any set of axioms but instead as vector of equilibrium payoffs of positive models of politics.

2.1 Lobbying and Power : The Nucleolus

In this section, we show that the nucleolus and more generally, the vectors belonging to the least core of the simple game arise as the vectors of equilibrium payoffs of a game describing the competition between two lobbies to buy the influence of the members of a legislature. More precisely, in Young (1978 a, b), Le Breton and Zaporozhets (2010) and Le Breton, Sudhölter and Zaporozhets (2010), it is shown that the least core and the nucleolus are in one to one correspondence with the set of vector of equilibrium payoffs of the legislators in a celebrated game of lobbying due to Groseclose and Snyder (1996) and further analysed by Banks (2000) and Diermeier and Myerson (1999). In this game theoretical model of lobbying, the players of the simple game are the legislators or public decision makers in charge of public policy. The legislators are assumed to be reactive to the influence of two lobbies and the public policy can be biased towards one side or the other depending upon the strength of each lobby and one key parameter characterizing the simple game and called the *hurdle factor* of the simple game. Le Breton and Zaporozhets (2010) and Le Breton, Sudhölter and Zaporozhets (2010) show how to calculate the hurdle factor.

As emphasized by Young, the nucleolus $NU (N, \mathcal{W})$ of the simple game $(N, \mathcal{W})$ can be interpreted as the vector of relative prices of the legislators' votes that a lobby has to pay to impose its most preferred outcome in the presence of the opposition. It can be shown that those prices are the solutions (up to a normalization) to the following linear program[13]:

$$\begin{aligned} &\min \sum_{i \in N} t_i \\ &\text{s.t.} \sum_{i \in S} t_i \geq 1 \text{ for all } S \in \mathcal{W} \; \cdot \\ &\quad t_i \geq 0 \text{ for all } i \in N \end{aligned} \tag{1}$$

It is important to point out that the set of prices that we obtain when blocking coalitions are considered differ from the set of prices when winning considered as above. The corresponding vector of prices are the solutions to the following linear program :

[12] For the definitions and the properties see for example, Felsenthal and Machover (1998) and Laruelle and Valenciano (2008).

[13] In fact, the equilibrium offers t_i coinside with the least core of the corresponding cooperative game. It may contain multiple solutions, but the nucleolus is always one of them.

$$\min \sum_{i \in N} t_i$$
$$\text{s.t. } \sum_{i \in S} t_i \geq 1 \text{ for all } S \in \mathcal{B} \cdot$$
$$t_i \geq 0 \text{ for all } i \in N$$

Therefore, in contrast to the measures of power advocated by Banzhaf and Shapley-Shubik which are invariant to the duality operation i.e. $BZ(N, \mathcal{W}) = BZ(N, \mathcal{B})$ and $SS(N, \mathcal{W}) = SS(N, \mathcal{B})$. In contrast, $NU(N, \mathcal{W}) \neq NU(N, \mathcal{B})$ except in the case where $(N, \mathcal{W})$ is constant sum. The second vector arises as a vector of equilibrium payoffs when the order of play of the two lobbies is inverted.

2.2 Bargaining and Power : The Nucleolus (Again)

In this section, we describe the power of the players as the payoffs they should expect to derive at equilibrium if the division of the pie proceeds from a legislative bargaining game constrained by some protocol. The game that we consider is the popular bargaining model introduced by Baron and Ferejohn (1989). The voting rule represented by a simple voting game $(N, \mathcal{W})$.

Bargaining proceeds as follows. At every round $t = 1, 2,$...Nature selects a random proposer: player i is selected with probability p_i). This player proposes a distribution of the budget $(x_1, ..., x_n)$ with $x_j \geq 0$ for all $j = 1, ...n$ and $\sum_{j=1}^{n} x_j = 1$. The proposal is voted upon immediately (closed rule). If the coalition of voters in favor of the proposal is winning, the proposal is implemented and the game ends; otherwise the game proceeds to the next period in which Nature selects a new proposer. Players are risk-neutral and discount future payoffs. The players discount the future payoffs by a factor $\delta_i \in [0, 1]$. A (pure) strategy for player i is a sequence $\sigma_i = (\sigma_i^t)\, t \geq 1$ where σ_i^t, the t-th round strategy of player i prescribes:

1. A proposal x.
2. A response function assigning "yes" or "no" to all possible proposals by the other players.

The solution concept is *stationary subgame perfect equilibrium* (SSPE). Stationarity requires that players follow the same strategy at every round t regardless of past offers and responses to past offers. Banks and Duggan (2000) have shown that an SSPE[14] always exists[15] in this type of bargaining model. In addition, Eraslan and McLennan (2006)[16] have shown that all SSPE lead to the same expected equilibrium payoffs.

[14] The main predictions of the model are the following. First, there is a property of immediate agreement. Even without discounting there is a pressure to reach agreement in the first period because of the risk of being excluded afterwards. Second, only minimal winning coalitions form in equilibrium, since otherwise it would be a waste of resources for the agenda setter. Third, the proposer receives a disproportional share of the pie, because he always buys the cheapest minimal coalition and pays the minimum amount to its members just to secure the acceptance of the proposal.

[15] The existence result is provided by Banks and Duggan (2000) in a very general setting in which the space of outcomes can be any convex compact set and the utility functions are concave but otherwise unrestricted.

[16] In the case of the standard majority game, the result was proved in Eraslan (2002).

In the case where $p_i = \frac{1}{n}$ and $\delta_i = 1$ for all $i = 1, ..., n$, we denote by $BF(N, \mathcal{W})$ the unique vector of equilibrium payoffs attached to the SSPE of the bargaining game. Hereafter, we will refer to this vector as the Baron-Ferejohn measure of power attached to the simple game $(N, \mathcal{W})$. To the best of our knowledge, Montero (2007) is the first to introduce and study this measure of power .

Montero (2006) has analysed the above bargaining game in the case where $(N, \mathcal{W})$ is a constant-sum homogeneous weighted majority game. She shows that when $p_i = \omega_i$ and $\delta_i \leq 1$ for all $i = 1, ..., n$ where ω denotes the unique homogeneous normalized representation of $(N, \mathcal{W})$, then the vector of equilibrium payoffs coincides with the nucleolus. More generally, for any proper simple game, she shows that if the vector p belongs to the least core of $(N, \mathcal{W})$, then there is a vector of equilibrium payoffs wich coincides with p. Finally, under some extra qualification on the simple game, she demonstrates that if the vector p belongs to the least core of $(N, \mathcal{W})$, then p is the unique vector of equilibrium payoffs. In her terminology, the nucleolus is a *self-confirming* measure of power.

3 Five Voting Bodies: Descriptive Analysis of Power

This section is purely descriptive. We analyze five weighted majority voting games associated to the Council of Ministers of the European Union in 1958, 1973, 1981, 1986 and 1995 (Table 1 is adapted from Felsenthal and Machover, 2001), and compare the distribution of the decision power according to the four different power measures.

We provide values for the Banzhaf and the Shapley-Shubik indices (calculated using the webpage of D. Leech) as well as an index obtained from the non-cooperative bargaining game due to Baron and Ferejohn (Montero, 2007) and the nucleolus. We are interested in power distribution in both approval and block situations. Both Banzhaf and Shapley-Shubik indices give the same answer, however, the other two measures may assign different capacity to approve or to block a proposal by a country.

3.1 Power Distribution in 1958

The European Community is represented by a weighted majority game $[12; 4, 4, 4, 2, 2, 1]$. As one can easily see Luxembourg is not in any winning or blocking coalition, and the game can be equivalently represented as $[6; 2, 2, 2, 1, 1, 0]$.

First, we look at the expected equilibrium payoffs in the bargaining game with the equal probabilities of being a proposer. We compare the results from Montero (2007) and the expected payoffs assuming that instead of the winning coalitions, we consider the blocking coalitions i.e. the dual game. Denote by x, y and z respectively the expected payoffs for players of type 2, 1 and 0. We impose $x = y$, then the equilibrium strategies might be summarized as follows:

.

The equations for the players' expected payoff then become:

Table 1: **Weights and quota in the Council of Ministers.**

Country	**1958**	**1973**	**1981**	**1986**	**1995**
Germany	4	10	10	10	10
Italy	4	10	10	10	10
France	4	10	10	10	10
UK	–	10	10	10	10
Spain	–	–	–	8	8
Belgium	2	5	5	5	5
Netherlands	2	5	5	5	5
Greece	–	–	5	5	5
Portugal	–	–	–	5	5
Sweden	–	–	–	–	4
Austria	–	–	–	–	4
Denmark	–	3	3	3	3
Ireland	–	3	3	3	3
Finland	–	–	–	–	3
Luxembourg	1	2	2	2	2
Quota	12	41	45	54	62
Total votes	17	58	63	76	87
Quota (%)	70.59	70.69	71.43	71.05	71.26

		Player type		
		x	y	z
Coalition type	$[2,2]$	$\frac{\lambda}{2}(2)$	–	$\frac{\gamma}{3}(3)$
	$[2,1]$	$\frac{1-\frac{\lambda}{2}}{2}(2)$	$1(1)$	$\frac{1-\frac{\gamma}{3}}{6}(6)$

$$
\begin{aligned}
x &= \frac{1}{6}(1-x) + \frac{2}{6}\frac{1}{3}x + \frac{1}{6}\left(\frac{2}{3}\gamma + \frac{1-\gamma/3}{6}\right)x + \frac{2}{6}\frac{\lambda}{2}x \\
y &= \frac{1}{6}(1-x) + \frac{3}{6}\frac{1-\lambda/2}{2}y + \frac{1}{6}\frac{1-\gamma/3}{6}y \\
z &= \frac{1}{6}(1-2x) \\
x &= y
\end{aligned}
$$

The solution is: $x = y = \frac{5}{28} \approx 0.179$, $z = \frac{3}{28} \approx 0.107$.

Interestingly, the medium-size countries get disproportionately high power as compared to the large ones, and the small country gets disproportionately high power as compared to the medium-size countries. The reason is that the small the medium countries have disproportionately high proposal power: the probability of being selected as a proposer is the same for all the countries and equals 1/6. Another interesting observation is that even though Luxembourg is a dummy, it gets positive expected payoff because it is allowed to make proposals.

In order to calculate the nucleolus (NU) we solve the problem (1) which looks like:

$$\begin{array}{l} \min 3x + 2y \\ \text{s.t. } 2x + 2y \geq 1 \\ \quad 3x \geq 1 \\ \quad x, y \geq 0 \end{array} .$$

The solution of this problem is:

$$x = \frac{1}{3},\ y = \frac{1}{6},$$

and the value of the program (the hurdle factor) is $\gamma = 1.333$.

If we look at the game with respect to the blocking coalitions, the nucleolus is the solution of the following program:

$$\begin{array}{l} \min 3x + 2y \\ \text{s.t. } x + y \geq 1 \\ \quad 2x \geq 1 \\ \quad x, y \geq 0 \end{array} .$$

.

The solution now is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the value of the program (the dual hurdle factor) is $\gamma = 2.5$.

The results are summarize in the following Table 2:

Table 2: **Power distribution in 1958.**

Country	**SS**	**BZ**	**BF**	**BF (b)**	**NU** $\gamma = 1.333$	**NU(b)** $\gamma = 2.5$
Germany	0.233	0.238	0.238	0.179	0.250	0.200
Italy	0.233	0.238	0.238	0.179	0.250	0.200
France	0.233	0.238	0.238	0.179	0.250	0.200
Netherlands	0.150	0.143	0.119	0.179	0.125	0.200
Belgium	0.150	0.143	0.119	0.179	0.125	0.200
Luxembourg	0	0	0.048	0.107	0	0

3.2 Power Distribution in 1973

The voting body is represented by the following weighted majority game: [41; 10, 10, 10, 10, 5, 5, 3, 3, 2]. There are 5 types of minimal blocking coalitions: [10, 10], [10, 5, 5], [10, 5, 3], [10, 3, 3, 2] and [5, 5, 3, 3, 2].

Again, we are looking for the expected equilibrium payoffs with respect to the blocking, and as before we denote the expected payoffs by x for the countries with 10 votes, by y - for the countries with 5 votes, by z- for the countries with 3 votes and by w - for the countries with 2 votes. We postulate an equilibrium with $y = z$ and $x < 2y$. The equilibrium strategies might be summarized as follows:

		Player type			
		x	y	z	w
Coalition type	[10, 10]	$\frac{1}{3}(3)$	$-$	$-$	$-$
	[10, 5, 5]	$-$	$\lambda(4)$	$-$	$-$
	[10, 5, 3]	$-$	$\frac{1-4\lambda}{8}(8)$	$\frac{1}{8}(8)$	$-$
	[10, 3, 3, 2]	$-$	$-$	$-$	$\frac{1}{4}(4)$

In the table we indicate the probability of proposing a particular coalition by a particular player, and in the parentheses we put the number of coalitions of a particular type that include a particular player.

The equations for the expected payoffs are given by:

$$\begin{aligned}
x &= \frac{1}{9}(1-x) + \frac{2}{9}\left(\lambda + 2\frac{1-4\lambda}{8}\right)x + \frac{2}{9}\frac{2}{8}x + \frac{1}{9}\frac{1}{4}x + \frac{3}{9}\frac{1}{3}x \\
y &= \frac{1}{9}(1-x-y) + \frac{2}{9}\frac{4}{8}y + \frac{1}{9}4\lambda y \\
z &= \frac{1}{9}(1-x-y) + \frac{2}{9}\frac{4(1-4\lambda)}{8}z + \frac{1}{9}z \\
w &= \frac{1}{9}(1-x-2z) \\
y &= z.
\end{aligned}$$

The unique expected equilibrium payoffs are:

$$x = \frac{4}{31} \approx 0.129,\ y = z = \frac{54}{527} \approx 0.102,\ w = \frac{39}{527} \approx 0.074.$$

Surprisingly, expected payoffs for countries with 5 and 3 votes are the same and do not differ much from the expected payoffs for the countries with 10 votes.

To calculate the nucleolus we solve the linear program:

$$
\begin{aligned}
\min\ & 4x + 2y + 2z + w \\
\text{s.t.}\ & 4x + y \geq 1 \\
& 4x + z \geq 1 \\
& 4x + w \geq 1 \\
& 3x + 2y + z \geq 1 \\
& 3x + 2y + w \geq 1 \\
& 3x + y + 2z \geq 1 \\
& x, y, z, w \geq 0
\end{aligned}
.
$$

The solution is $(1/3, 0, 0, 0)$ and the value of the program is $4/3$. As compared to 1958 the hurdle factor does not change, as well as the power of the big countries. However, other countries, even though they are not dummies, get zero.

Looking at the minimal blocking coalitions we need to solve:

$$
\begin{aligned}
\min\ & 4x + 2y + 2z + w \\
\text{s.t.}\ & x + y + z \geq 1 \\
& x + 2y \geq 1 \\
& 2x \geq 1 \\
& 2y + 2z + w \geq 1 \\
& x + 2z + w \geq 1 \\
& x, y, z, w \geq 0
\end{aligned}
$$

We deduce that the nucleolus in this case is $\left(\frac{1}{6}, \frac{1}{12}, \frac{1}{12}, 0\right)$ and the dual hurdle factor is $\gamma = 3$. It is interesting to notice, that even though Luxembourg is not a dummy anymore it gets 0. Further, the hurdle factor is increasing as compared to the previous case, which means that the Council became less vulnerable to lobbying.

The results are summarized in the Table 3.

Table 3: **Power distribution in 1973.**

Country	**SS**	**BZ**	**BF**	**BF (b)**	**NU** $\gamma = 1.333$	**NU (b)** $\gamma = 3.0$
Germany	0.179	0.167	0.159	0.129	0.250	0.167
Italy	0.179	0.167	0.159	0.129	0.250	0.167
France	0.179	0.167	0.159	0.129	0.250	0.167
UK	0.179	0.167	0.159	0.129	0.250	0.167
Belgium	0.081	0.091	0.079	0.102	0	0.083
Netherlands	0.081	0.091	0.079	0.102	0	0.083
Denmark	0.057	0.066	0.071	0.102	0	0.083
Ireland	0.057	0.066	0.071	0.102	0	0.083
Luxembourg	0.001	0.016	0.063	0.074	0	0

3.3 Power Distribution in 1981

As it is shown in Montero (2007) the representation $[45; 10, 10, 10, 10, 5, 5, 5, 3, 3, 2]$ is equivalent to $[18; 4, 4, 4, 4, 2, 2, 2, 1, 1, 1]$.

The nucleolus is the solution of the linear program:

$$\begin{aligned} &\min 4x + 3y + 3z \\ &\text{s.t. } 4x + y \geq 1 \\ &\quad 4x + 2z \geq 1 \\ &\quad 3x + 3y \geq 1 \\ &\quad 3x + 2y + 2z \geq 1 \\ &\quad x, y, z \geq 0 \end{aligned}$$

The minimum is reached at $(1/3, 0, 0)$, and the value of this minimum is $4/3$. In fact, nothing is changed as compared to 1973.

The following linear program:

$$\begin{aligned} &\min 4x + 3y + 3z \\ &\text{s.t. } 2x \geq 1 \\ &\quad x + 2y \geq 1 \\ &\quad x + y + 2z \geq 1 \\ &\quad 3y + 2z \geq 1 \\ &\quad x, y, z \geq 0 \end{aligned}$$

gives the solution if we are interested in the game with respect to blocking situation. The nucleolus in this case is $(0.16, 0.08, 0.04)$ and the dual hurdle factor is $\gamma = 3.125$.

When calculating the expected payoffs in the case the blocking coalitions are proposed, we checked that the assumption made in Montero, 2007:

$$x = 2y \text{ and } y = 2z$$

is not supported in any equilibrium anymore. We impose an equilibrium with

$$x < 2y \text{ and } y < 2z.$$

Then, the equilibrium strategies can be summarized in the following table:

		Player type		
		x	y	z
Coalition type	$[4, 4]$	$\frac{1}{3}(3)$	–	–
	$[4, 2, 2]$	–	$\frac{1}{8}(8)$	–
	$[4, 2, 1, 1]$	–	–	$\frac{1}{24}(24)$
	$[2, 2, 2, 1, 1]$	–	–	–

The system for the equilibrium expected payoffs is:

$$
\begin{aligned}
x &= \frac{1}{10}(1-x) + \frac{3}{10}\frac{2}{8}x + \frac{3}{10}\frac{6}{24}x + \frac{3}{10}\frac{1}{3}x \\
y &= \frac{1}{10}(1-x-y) + \frac{3}{10}\frac{8}{24}y + \frac{2}{10}\frac{4}{8}y \\
z &= \frac{1}{10}(1-x-y-z) + \frac{2}{10}\frac{12}{24}z.
\end{aligned}
$$

The solution is:

$$x = 0.118,\ y = 0.098 \text{ and } z = 0.078$$

Interestingly, $x < 2y$ and $y < 2z$, i.e. the price per vote of a smaller player is higher than for a bigger one.

The results are summarized in the Table 4.

Table 4: **Power distribution in 1981.**

Country	**SS**	**BZ**	**BF**	**BF (b)**	**NU** $\gamma = 1.333$	**NU (b)** $\gamma = 3.125$
Germany	0.174	0.158	0.160	0.118	0.250	0.160
Italy	0.174	0.158	0.160	0.118	0.250	0.160
France	0.174	0.158	0.160	0.118	0.250	0.160
UK	0.174	0.158	0.160	0.118	0.250	0.160
Belgium	0.071	0.082	0.080	0.098	0	0.080
Netherlands	0.071	0.082	0.080	0.098	0	0.080
Greece	0.071	0.082	0.080	0.098	0	0.080
Denmark	0.030	0.041	0.040	0.078	0	0.040
Ireland	0.030	0.041	0.040	0.078	0	0.040
Luxembourg	0.030	0.041	0.040	0.078	0	0.040

3.4 Power Distribution in 1986

The game is described as $[54; 10, 10, 10, 10, 8, 5, 5, 5, 5, 3, 3, 2]$. With respect to the blocking the game can be written as: $[23; 10, 10, 10, 10, 8, 5, 5, 5, 5, 3, 3, 2]$. By ω we denote the number of minimum winning coalitions.

Table 5: **Power distribution in 1986.**

Country	**SS**	**BZ**	**NU** $\gamma = 1.38$ $\omega = 135$	**NU (b)** $\gamma = 3.2$ $\omega = 182$
Germany	0.134	0.129	0.138	0.125
Italy	0.134	0.129	0.138	0.125
France	0.134	0.129	0.138	0.125
UK	0.134	0.129	0.138	0.125
Spain	0.111	0.109	0.103	0.125
Belgium	0.064	0.067	0.069	0.063
Netherlands	0.064	0.067	0.069	0.063
Greece	0.064	0.067	0.069	0.063
Portugal	0.064	0.067	0.069	0.063
Denmark	0.043	0.046	0.034	0.063
Ireland	0.043	0.046	0.034	0.063
Luxembourg	0.012	0.018	0	0

3.5 Power Distribution in 1995

The game is described as $[62; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]$ with total weight 87. With respect to the blocking the game becomes: $[26; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]$.

4 Qualified Majority Voting

4.1 QMV in non-enlarged CM

$\mathcal{W}_{15} = [169; 29, 29, 29, 29, 27, 13, 12, 12, 12, 10, 10, 7, 7, 7, 4]$

$\mathcal{P}_{15} = [2327; 820, 592, 590, 576, 394, 158, 105, 102, 100, 89, 81, 53, 52, 37, 4]$ (total weight is 3753) is the weighted rule whose weights are population sizes of 15 countries and quota is 62%. The following Table 7 presents the results.

4.2 QMV in a 27-member CM

Following Felsenthal and Machover (2001) and Bilbao et al. (2002) we consider different variants.

The first variant is a double majority system $v_1 \cap v_2$, or $v_1 \cap v_3$. The rule v_1 is either the weighted rule with weighted votes described by

Table 6: **Power distribution in 1995.**

Country	**SS**	**BZ**	**NU** $\gamma = 1.4$ $\omega = 829$	**NU (b)** $\gamma = 3.33$ $\omega = 1270$
Germany	0.117	0.112	0.115	0.1
Italy	0.117	0.112	0.115	0.1
France	0.117	0.112	0.115	0.1
UK	0.117	0.112	0.115	0.1
Spain	0.095	0.092	0.092	0.1
Belgium	0.056	0.059	0.057	0.05
Netherlands	0.056	0.059	0.057	0.05
Greece	0.056	0.059	0.057	0.05
Portugal	0.056	0.059	0.057	0.05
Sweden	0.045	0.048	0.046	0.05
Austria	0.045	0.048	0.046	0.05
Denmark	0.035	0.036	0.034	0.05
Ireland	0.035	0.036	0.034	0.05
Finland	0.035	0.036	0.034	0.05
Luxembourg	0.021	0.023	0.023	0.05

$\mathcal{W}_{27} = [255; 29, 29, 29, 29, 27, 27, 14, 13, 12, 12, 12, 12, 12, 10, 10, 10, 7, 7, 7, 7, 7, 4, 4, 4, 4, 4, 3]$[17].

The rule v_2 is rule $\mathcal{P}_{27}$, the weighted rule whose weights are population sizes of 27 members and whose quota is equal to 62%:

$\mathcal{P}_{27} = [620; 170, 123, 122, 120, 82, 80, 47, 33, 22, 21, 21, 21, 21, 18, 17, 17, 11, 11, 11, 8, 8, 5, 4, 3, 2, 1, 1]$.

Finally, v_3 is either $\mathcal{M}_{27}$, the ordinary majority rule, with weight 1 for each country and quota 14, or $\mathcal{M}'_{27}$, the ordinary majority rule, with weight 1 for each country and quota 18 :

$\mathcal{M}_{27} = [14; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$;

$\mathcal{M}'_{27} = [18; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$;

The second variant is a triple majority system $v_1 \cap v_2 \cap v_3$, where v_1, v_2 and v_3 are as before.

In the following table we report the number of minimum winning coalitions for each analyzed rule:

One can notice that there is not a big difference in terms of the number of the minimum winning coalitions between $\mathcal{W}_{27}$, $\mathcal{W}_{27} \cap \mathcal{P}_{27}$, $\mathcal{W}_{27} \cap \mathcal{M}_{27}$ and $\mathcal{W}_{27} \cap \mathcal{M}_{27} \cap \mathcal{P}_{27}$ or between $\mathcal{W}_{27} \cap \mathcal{M}'_{27}$ and $\mathcal{W}_{27} \cap \mathcal{M}'_{27} \cap \mathcal{P}_{27}$.

Interestingly, the hurdle factor γ is not affected by the additional requirements and it

[17] Sometimes in the literature quota 258 is used because of the discrepancies in the Nice Treaty. It appears that the correct number is 255. However, we perform calculations also for quota 258, and we did not find significant differences.

Table 7: **Power distribution for the 15 EU countries under the nucleolus.**

Country	$\mathcal{W}_{15}$		$\mathcal{W}_{15} \cap \mathcal{P}_{15}$	
	NU	**NU(b)**	**NU**	**NU(b)**
	$\gamma = 1.4$	$\gamma = 3.414$	$\gamma = 1.4$	$\gamma = 3.483$
	$\omega = 775$	$\omega = 1018$	$\omega = 760$	$\omega = 1490$
Germany	0.122	0.121	0.122	0.139
Italy	0.122	0.121	0.122	0.119
France	0.122	0.121	0.122	0.119
UK	0.122	0.121	0.122	0.119
Spain	0.112	0.111	0.112	0.109
Belgium	0.051	0.061	0.051	0.059
Netherlands	0.051	0.051	0.051	0.050
Greece	0.051	0.051	0.051	0.050
Portugal	0.051	0.051	0.051	0.050
Sweden	0.041	0.040	0.041	0.050
Austria	0.041	0.040	0.041	0.040
Denmark	0.031	0.030	0.031	0.030
Ireland	0.031	0.030	0.031	0.030
Finland	0.031	0.030	0.031	0.030
Luxembourg	0.020	0.020	0.020	0.020

remains the same ($\gamma = 1.337$) for all combinations. The nucleolus also assigns the same values under all these rules. The results are given in the subsequent Table 9.

Table 8: **Power distribution for the 27 EU countries according to the nucleolus under different rules.**

rule	ω
$\mathcal{W}_{27}$	476063
$\mathcal{W}_{27} \cap \mathcal{P}_{27}$	476060
$\mathcal{W}_{27} \cap \mathcal{M}_{27}$	476063
$\mathcal{W}_{27} \cap \mathcal{M}'_{27}$	684204
$\mathcal{W}_{27} \cap \mathcal{M}_{27} \cap \mathcal{P}_{27}$	476060
$\mathcal{W}_{27} \cap \mathcal{M}'_{27} \cap \mathcal{P}_{27}$	684201

Table 9: **Power distribution for the 27 EU countries according to the nucleolus under different rules.**

Country	**NU**
Germany	0.084
UK	0.084
France	0.084
Italy	0.084
Spain	0.078
Poland	0.078
Romania	0.041
Netherlands	0.038
Greece	0.035
Czech Republic	0.035
Belgium	0.035
Hungary	0.035
Portugal	0.035
Sweden	0.029
Bulgaria	0.029
Austria	0.029
Slovak Republic	0.020
Denmark	0.020
Finland	0.020
Ireland	0.020
Lithuania	0.020
Latvia	0.012
Slovenia	0.012
Estonia	0.012
Cyprus	0.012
Luxembourg	0.012
Malta	0.009

5 The Power of the European Citizens and the Optimal Decision Rule

In the previous sections we calculated the power of each nation (representative) in the Council of Ministers of the European Union measured for four measures of power. In this section, we will focus on the nucleolus and we will adopt a normative perspective. As already explained, focusing on the nucleolus simply means that we are interested by European policy issues which can be described formally as distributive politics. Something has to be shared among the members of the council of ministers and ultimately among the European citizens and the nucleolus is the reduced form of equilibrium for several alternative game forms spanning bargaining and lobbying. To fix ideas, let us for the time being interpret this pie as the gains (measured in appropriate units) resulting from European coordination. Fairness suggests to allocate these gains equally across European citizens. This means that each country should receive a share proportional to its population size. If there were no intermediate political bodies i.e. if the simple game to be considered was the majority game with the set of European citizens as the set of players, then all the coordinates of the nucleolus would be equal and proportionality would be fulfilled. Unfortunately, we are in a second best environment : the negotiation takes place across the countries. Only, in a second stage, the share obtained by each country is divided among the citizens of the country. We are left with a non trivial mechanism design exercise because we need to evaluate the citizens' indirect power via their representatives in a two-stage decision-making process: at the first stage citizens elect their representative (exercise their direct power), and at the second stage the representative make an actual decision (citizens exercise only indirect power).

In what follows, we use a similar approach as in Felsenthal and Machover (1998) to measure citizens' indirect Banzhaf power in a two-tier system. Their main result is that citizens' indirect Banzhaf powers are equal if and only if the Banzhaf powers of the delegates in the council are proportional to the respective square root of the population size[18]. Algabada et al. (2007) apply this theory to analyze the power of the European citizens for 25 and 27 countries. In the proposition below we prove a similar theoretical result for the relative voting power measured by the nucleolus and then apply it to the Council of Ministers of the European Union.

To describe the two-stage political process we use the following notations. Let the simple voting game $\Gamma_0 = (M, \mathcal{W}^0)$ describes the decision-making process at the council, where $M = \{1, ..., m\}$ is the set of countries and $\mathcal{W}^0$ is set of all winning coalitions. Also, by u we denote the characteristic function. Similarly, the game $\Gamma_i = (N_i, \mathcal{W}^i)$, $i = 1...m$ refers to the decision-making process for each country i. Naturally, we assume that the sets N_i are disjoint. Then, the *compound* game $\Gamma = \Gamma_0 [\Gamma_1, ..., \Gamma_m]$ is defined over set $N = N_1 \cup ... \cup N_m$ and its characteristic function v is defined by

$$v(S) = u(\{i \in M : S \cap N_i \in \mathcal{W}^i\}),\ S \subset N.$$

We also denote by n_i and n the size of N_i and N respectively. We adopt an assumption

[18] See their theorem 3.43.

from Felsenthal and Machover (1998) that each component Γ_i is a quota majority game with the same quota[19] $q \geq 1/2$ for all $i = 1...m$. By assumption, the numbers n_i are very large.

Proposition 1 *The nucleolus ν of the simple game Γ can be expressed through the nucleolus ν^0 of the game Γ_0 as follows*

$$\nu = \left(\underbrace{\frac{\nu_1^0}{n_1}, ..., \frac{\nu_1^0}{n_1}}_{n_1 \ times}, ..., \underbrace{\frac{\nu_m^0}{n_m}, ..., \frac{\nu_m^0}{n_m}}_{n_m \ times} \right). \tag{2}$$

Proof. In order to find the nucleolus for the game Γ (up to a normalization) we need to solve the following linear minimization problem:

$$\begin{aligned} &\min \sum_{i \in M} \sum_{j \in N_i} t_{ij} \\ &\text{s.t.} \sum_{i \in S} \sum_{j \in T_i} t_{ij} \geq 1 \text{ for } T_i \in \mathcal{W}^i, S \in \mathcal{W}^0 \ . \\ &\quad t_{ij} \geq 0 \text{ for } i \in M, j \in N_i \end{aligned} \tag{3}$$

The numbers t_{ij} reflect the amounts each citizen j in country i gets. Without loss of generality we can take $t_{ij} = t_i$, i.e. the citizens' of country i get the same amount. Then, the problem (3) can be rewritten as follows:

$$\begin{aligned} &\min \sum_{i \in M} n_i t_i \\ &\text{s.t.} \sum_{i \in S} q n_i t_i \geq 1 \text{ for } S \in \mathcal{W}^0 \ . \\ &\quad t_i \geq 0 \text{ for } i \in M \end{aligned}$$

Applying the substitution for $t_i' = q n_i t_i$ the minimization problem equivalently can be rewritten as:

$$\begin{aligned} &\min \sum_{i \in M} t_i' \\ &\text{s.t.} \sum_{i \in S} t_i' \geq 1 \text{ for } S \in \mathcal{W}^0 \ . \\ &\quad t_i' \geq 0 \text{ for } i \in M \end{aligned} \tag{4}$$

One can notice that the final problem (4) is the problem for the representatives. Therefore, we proved that $t_i = \frac{1}{q n_i} t_i'$ and taking into account normalization we establish the claim. ■

From the proof of the proposition it also follows that the hurdle factor γ of the compound game is equal to the hurdle factor γ^0 of the game for the representatives multiplied by $\frac{1}{q}$. The determination of the nucleolus of a compound simple game is not straightforward. Our

[19] In fact, Felsenthal and Machover (1998) assume that the components are simple majority games, i.e. $q = 1/2$.

proposition is a specific case of a more general result by Megiddo (1971, 1974). He shows that the nucleolus ν of a compound game Γ can be expressed as follows:

$$\nu = \alpha_1 \nu^{1*} + ... + \alpha_m \nu^{m*},$$

where ν^{i*} is a baricentrical projection of ν^i on N_i, i.e.

$$\nu_j^{i*} = \begin{cases} \nu_j^i, & j \in N_i \\ 0, & j \notin N_i \end{cases}$$

and α_i is a solution to an optimization problem[20]. In our case $\alpha = \nu^0$.

Corollary 2 *Citizens' indirect powers measured by the nucleolus ν_i are equal for all $i \in N$ iff the powers of the delegates ν_j^0 are equal to the respective population rates $\frac{n_j}{n}$.*

The optimization variable is here the simple game $(M, \mathcal{W}^0)$. There is a finite number of possible choices. This number can be large in particular if we do not impose any restrictions on the nature of the simple game itself. In appendix 3, we have reported some results from the literature on the enumeration of all simple games or important families of simple games. On top of these families, appear the family of (strong) weighted majority games. If we limit the optimization to that subclass, then we may think of using Peleg's result asserting that the normalized homogeneous representation of a homogeneous strong weighted majority game $(N, \mathcal{W})$ coincides with the nucleolus of (N, v). If the game generated by the weights $\omega_i = n_i$ and the quota $\frac{\sum_{i \in M} \omega_i}{2}$ is homogeneous, then the solution of our problem is trivial as we can get the first best. Unfortunately, things are much less simple. In what follows, we will formulate the combinatorial optimization problem that we consider and derive the optimal simple game $(M, \mathcal{W}^0)$. Before doing so, it is useful to evaluate how the implications of the choices of $(M, \mathcal{W}^0)$ on the nucleolus for the five stages of European enlargement which are considered in this paper In the following two tables 10 and 11 we show the population ratios taken from Felsenthal and Machover (1998, 2004) and the nucleolus taken from the tables in the previous section. An asterisk indicates an occurrence of the paradox of new members: a member state's relative power has increased although its relative weight has decreased as a result of the accession of the new members. One can notice that it happens for example, in 1995 when Luxembourg gains in relative power from 0 to 0.023.

[20] See his Theorem 5.6.

Table 10: **Population and the nucleolus in the Council of Ministers 1958-1995.**

Country	1958		1973		1981		1986		1995	
	$\frac{n_j}{n}$	**NU**	$\frac{n_j}{n}$	**NU**	$\frac{n_j}{n}$	**NU**	$\frac{n_j}{n}$	**NU**	$\frac{n_j}{n}$	**NU**
France	0.266	**0.250**	0.203	**0.250**	0.200	**0.250**	0.172	**0.138**	0.156	**0.115**
Germany	0.322	**0.250**	0.242	**0.250**	0.228	**0.250**	0.189	**0.138**	0.220	**0.115**
Italy	0.291	**0.250**	0.214	**0.250**	0.209	**0.250**	0.176	**0.138**	0.154	**0.115**
Belgium	0.053	**0.125**	0.038	**0**	0.036	**0**	0.031	**0.069***	0.027	**0.057**
Netherlands	0.066	**0.125**	0.052	**0**	0.053	**0**	0.045	**0.069***	0.042	**0.057**
Luxembourg	0.002	**0**	0.001	**0**	0.001	**0**	0.001	**0**	0.001	**0.023***
UK	–	–	0.218	**0.250**	0.205	**0.250**	0.176	**0.138**	0.157	**0.115**
Denmark	–	–	0.019	**0**	0.019	**0**	0.016	**0.034***	0.014	**0.034**
Ireland	–	–	0.012	**0**	0.013	**0**	0.011	**0.034***	0.010	**0.034**
Greece	–	–	–	–	0.036	**0**	0.031	**0.069***	0.028	**0.057**
Spain	–	–	–	–	–	–	0.120	**0.103**	0.105	**0.092**
Portugal	–	–	–	–	–	–	0.031	**0.069**	0.027	**0.057**
Austria	–	–	–	–	–	–	–	–	0.022	**0.046**
Sweden	–	–	–	–	–	–	–	–	0.024	**0.046**
Finland	–	–	–	–	–	–	–	–	0.014	**0.034**

Table 11: **Population and the nucleolus in the Council of Ministers under QM rules with 15 and 27 members.**

Country	QM 15		QM 27	
	$\frac{n_j}{n}$	NU	$\frac{n_j}{n}$	NU
Germany	0.219	**0.122**	0.170	**0.084**
France	0.157	**0.122**	0.123	**0.084**
UK	0.158	**0.122**	0.123	**0.084**
Italy	0.154	**0.122**	0.120	**0.084**
Spain	0.105	**0.112**	0.082	**0.078**
Poland	–	–	0.080	**0.078**
Romania	–	–	0.047	**0.041**
Netherlands	0.042	**0.051**	0.033	**0.038**
Greece	0.028	**0.051**	0.022	**0.035**
Portugal	0.027	**0.051**	0.021	**0.035**
Belgium	0.027	**0.051**	0.021	**0.035**
Czech Republic	–	–	0.021	**0.035**
Hungary	–	–	0.021	**0.035**
Sweden	0.024	**0.041**	0.018	**0.029**
Austria	0.022	**0.041**	0.017	**0.029**
Bulgaria	–	–	0.017	**0.029**
Denmark	0.014	**0.031**	0.011	**0.02**
Slovak Republic	–	–	0.011	**0.02**
Finland	0.014	**0.031**	0.011	**0.02**
Ireland	0.010	**0.031**	0.008	**0.02**
Lithuania	–	–	0.008	**0.02**
Latvia	–	–	0.005	**0.012**
Slovenia	–	–	0.004	**0.012**
Estonia	–	–	0.003	**0.012**
Cyprus	–	–	0.002	**0.012**
Luxembourg	0.001	**0.020**	0.001	**0.012**
Malta	–	–	0.001	**0.009**

Obviously, the results suggest that the European citizens are not treated equally under the decision rules operating in the CM since 1958 till now. The reason is that the nucleolus does not coincide with the population ratios, i.e. the corollary 2 is not satisfied. In what follows we investigate the question whether it were possible to do better and describe the methodology to choose the optimal decision rule.

6 The Optimal (Fair) Decision Rules

Corollary 2 suggests that if we would like to equalize the citizens' power under the nucleolus, we need to choose such a voting rule which will lead to the nucleolus ν_j^0 for the representatives being equal to the countries' population sizes. However, except in some exceptional circumstances, it is not always possible to find a game, for which the vector of countries' population sizes coincides with the nucleolus. Our tables provide information on the distance between the first best and the outcome of the choices which were made. These choices may be third best choices and we would like now to report on what could or should have the second best from the perspective of our nucleolus measure of benefit.

Hereafter, we will assume that the objective of the political architect is to design the simple game $(M, \mathcal{W}^0)$ in such a way that the distance between the induced nucleolus calculated at the citizen level and the first best is the smallest possible. The distance which is is considered here is the quadratic distance where the units are the citizens instead of the countries. The objective of minimizing the variance is peculiar; the minimization of any other inequality index like the Gini index or a Kolm-Atkinson index as reflecting the desire to meet an egalitarian norm would be very appropriate too. Maaser and Napel (2006) refer to this variance evaluation at the individual level as being the *cumulative individual quadratic deviation*. Beisbart and Bovens (2007) also use the quadratic criterion way to measure departure from perfect equality. While different, our approach follows the direction paved by Barbera and Jackson (2006) who consider instead an *utilitarian* criterion. This welfarist approach has been followed by several authors among which Biesbart, Bovens and Hartmann (2005) and Biesbart and Bovens (2007).

Denoting by $\mathcal{S}_m$ the set of all simple games with m players, our combinatorial problem is defined as follows:

$$\underset{(M,\mathcal{W}^0)\in\mathcal{S}_m}{Min} Var\left(NU\left((M,\mathcal{W}^0)\right)\right),$$

where

$$Var\left(NU\left((M,\mathcal{W}^0)\right)\right) = \sum_{i\in M} n_i \left[\frac{1}{n} - \frac{\nu_i^0}{n_i}\right]^2, \tag{5}$$

where $NU((M,\mathcal{W}^0)) = (\nu_1^0, \nu_2^0, ..., \nu_m^0)$. The term $\frac{\nu_i^0}{n_i}$ indicates how much power (according to the nucleolus) a citizen in country i gets given a specific voting rule. One can notice that (5) can be simplified as

$$Var\left(NU\left((M,\mathcal{W}^0)\right)\right) = \sum_{i\in M} \frac{\left(\nu_i^0\right)^2}{n_i} - \frac{1}{n}. \tag{6}$$

The resolution of our problem would be greatly simplified if we knew the image $Im\,(NU_m)$ of the mapping NU_m attaching to any simple game $(M,\mathcal{W}^0) \in \mathcal{S}_m$ the nucleolus of the game. $Im\,(NU_m)$ is a finite subset of the $(m-1)-$ dimensional simplex. If $Im\,(NU_m)$ was characterized, our problem would be:

$$\begin{array}{l} Min \sum_{i\in M} \frac{(x_i)^2}{n_i} \\ \text{s.t. } x \in Im\,(NU_m) \end{array}. \tag{7}$$

This formulation indicates quite clearly why the second best problem differs from its first analogue where the constraint $x \in Im\,(NU_m)$ is replaced by the relaxed constraint $x \in \left\{ y \in \mathbb{R}^m_+ : \sum_{i=1}^{m} y_i = 1 \right\}$. The first order conditions write:

$$\frac{2\nu_i^0}{n_i} = \lambda \text{ for } i \in M,$$

where λ is a Lagrange multiplier. From these conditions, we deduce that

$$\nu_i^0 = \frac{n_i}{n} \text{ for } i \in M,$$

which is as expected the egalitarian first best. Unfortunately, the set(s) has(ve) not been characterized in full generality. This problem, known as an inverse problem as the problem is to characterize which vectors can be obtained as a power vector for an adequate choice of a simple game, has been formulated recently by Alon and Edelman (2010) for the Banzhaf measure and they obtained partial results. We are not aware of any general result on the inverse problem for the nucleolus. This means that we will examine the combinatorial problem in its original formulation. Precisely, we consider as subset of feasible simple games any subset $\mathcal{G}_m$ of the set $\mathcal{S}_m$ of all simple games with a special focus on the set of constant sum simple games. However, any other subset of $\mathcal{S}_m$ like for instance, the set of weighted majority simple games or homogeneous weighted majority games or weighted majority games where the weights are constrained by some symmetry conditions could be considered as well. The procedure for solving (7) can be presented as the sequence of the following steps:

Step 1. For the given number of countries m, list all possible games the class $\mathcal{G}_m$;

Step 2. Calculate the nucleolus ν^0 for each game from the list;

Step 3. Find the variance from (6);

Step 4. Choose the game with the minimal variance.

We illustrate the use of our technique for $m = 3$ and 4. Without loss of generality, we assume that $n_1 \geq n_2 \geq ... \geq n_m$.

For 3 countries there are only two possible strong games: the simple majority game which is represented as $[2; 1, 1, 1]$ and the dictatorial game which is represented as $[1; 1, 0, 0]$. Then given (6) the variance for the majority game is:

$$Var_{maj} = \frac{1}{9}\left[\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right] - \frac{1}{n},$$

and the variance for the dictatorial game is:

$$Var_{dict} = \frac{1}{n_1} - \frac{1}{n}.$$

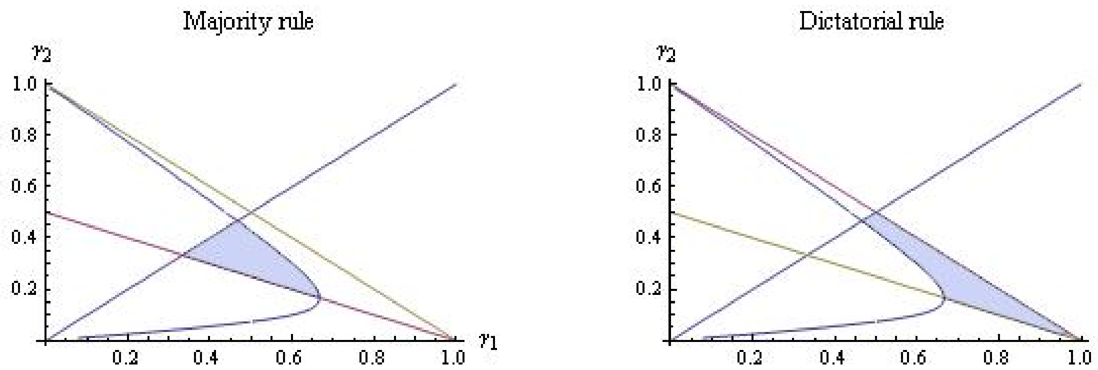

Figure 1: The optimal rule in a class of strong games for $m = 3$.

On the following graphs (figure 1) we show the values of the population shares γ_1 and γ_2 for the two biggest countries (where $\gamma_i = \frac{n_i}{n}$) for which each of the two games is optimal.

If we drop the restriction of strong, we have an additional game, where there are two vetoers[21] with the representation $[2; 1, 1, 0]$. The variance for such a rule is:

$$Var_{veto} = \frac{1}{4}\left[\frac{1}{n_1} + \frac{1}{n_2}\right] - \frac{1}{n}.$$

On the following figures 2 we again show the values of the two biggest countries' population shares, γ_1 and γ_2, for which each of the three games is optimal.

Not surprising the majority rule is optimal when the three countries are not too different in terms of the population ratios, and the dictatorial rule is optimal in the case where there is a relatively big country.

For 4 countries there are only 3 possible games in the class of strong games: $[1; 1, 0, 0, 0]$ (dictatorial rule), $[2; 1, 1, 1, 0]$ (majority rule for three players) and $[3; 2, 1, 1, 1]$ (apex game). As before the variance for the majority game is:

$$Var_{maj} = \frac{1}{9}\left[\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right] - \frac{1}{n},$$

and the variance for the dictatorial game is:

$$Var_{dict} = \frac{1}{n_1} - \frac{1}{n}.$$

The variance for the apex game is:

$$Var_{apex} = \frac{1}{25}\left[\frac{4}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right] - \frac{1}{n}.$$

[21] In fact, we could include the unanimmity game, but it gives the same variance as the simple majority game.

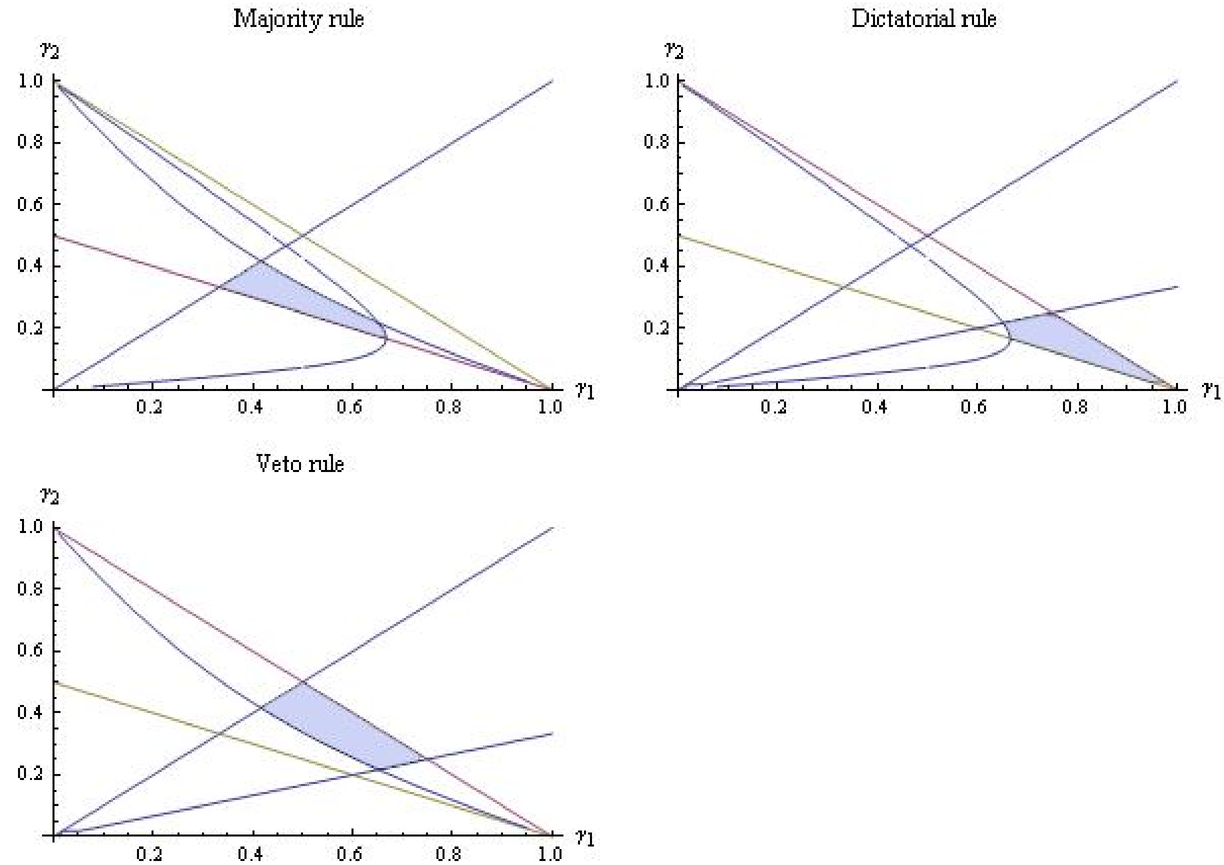

Figure 2: The optimal rule for $m = 3$.

Similarly to the previous case, the majority rule is optimal when the three countries are relatively close in the population size and the fourth country is very small.

From the results reported in appendix 2, it is clear that we can solve our optimality problem by "brutal force" as long as $m \leq 8$ for some specific important classes of simple games, like for instance, strong weighted majority games. For such games, we will limit (of course) to those where the weights are congruent to the population sizes. For $m \leq 8$, any weighted majority game admits a unique minimal integral representation which coincides with the least core and therefore the nucleolus. The relationship between representations and nucleolus starts to become intricate when $m \geq 9$. The nucleolus is always a representation but it does not always induce a minimal integral representation even in simple games with a unique such minimal integral representation.

We take advantage of this methodological detour to discuss briefly the issue of representation of weighted majority games which is discussed with more details in appendices 2 and 3. We know that the notion of weight is meaningless in the measurement of power as what matters exclusively is the structure of minimal winning coalitions resulting from the weights. However, in the literature on the design of optimal voting organizations, authors often refer to optimal weights which could be misleading as we could infer that the numerical values of the weights make sense. They raise questions, like for instance: should these weights vary like the population size or the square root of the population size or any other power of the population size? We think that if the design question has to be formulated in terms of representation, then it could be formulated as follows : among all potential minimal integral representations of strong weighted majority simple games, which one should be selected given the considered objective? At the extreme, suppose that the organization has only 3 members whose populations sizes satisfy $n_1 \geq n_2 \geq n_3$. Then the question of allocating voting weights to the countries according the to the values of n_i versus the values of $\sqrt{n_i}$ has no interest. A formulation using canonical minimal integral representations is instead very meaningful and it offers some extra advantages, like for instance, the knowledge of the seats' number (or total weight) necessary to proceed. Of course, when m is large the combinatorial issue becomes out of reach and it may be useful to see if then working with the n_i as opposed to the $\sqrt{n_i}$ lead to very contrasted simple games. It is then an empirical matter to determine what we mean by large m to move from the combinatorics to the calculation through simple functions of weights.

6.1 The EU Council of Ministers 1958 Revisited

In this final part we apply our technique to find the optimal decision rule for the EU Council of Ministers in 1958 given the number of member states and their population sizes (see Table 10). In the following Table 12 we list all strong weighted majority games as provided by Isbell (1959) with the corresponding values for the nucleolus and for $nVar$.

Table 12: **Strong Weighted Voting Games with 6 Players.**

	NU	nVar
$[1,0,0,0,0,0]$	$(1,0,0,0,0,0)$	**2.106**
$[1,1,1,0,0,0]$	$(1/3,1/3,1/3,0,0,0)$	**0.145**
$[2,1,1,1,0,0]$	$(2/5,1/5,1/5,1/5,0,0)$	**0.391**
$[1,1,1,1,1,0]$	$(1/5,1/5,1/5,1/5,1/5,0)$	**0.773**
$[3,1,1,1,1,0]$	$(3/7,1/7,1/7,1/7,1/7,0)$	**0.412**
$[2,2,1,1,1,0]$	$(2/7,2/7,1/7,1/7,1/7,0)$	**0.305**
$[3,2,2,1,1,0]$	$(1/3,2/9,2/9,1/9,1/9,0)$	**0.120**
$[2,1,1,1,1,1]$	$(2/7,1/7,1/7,1/7,1/7,1/7)$	**10.299**
$[4,1,1,1,1,1]$	$(4/9,1/9,1/9,1/9,1/9,1/9)$	**6.295**
$[3,2,1,1,1,1]$	$(1/3,2/9,1/9,1/9,1/9,1/9)$	**6.154**
$[4,2,2,1,1,1]$	$(4/11,2/11,2/11,1/11,1/11,1/11)$	**4.062**
$[3,3,2,1,1,1]$	$(3/11,3/11,2/11,1/11,1/11,1/11)$	**4.024**
$[4,3,3,1,1,1]$	$(4/13,3/13,3/13,1/13,1/13,1/13)$	**2.837**
$[5,2,2,2,1,1]$	$(5/13,2/13,2/13,2/13,1/13,1/13)$	**3.059**
$[5,3,3,2,1,1]$	$(1/3,1/5,1/5,2/15,1/15,1/15)$	**2.208**
$[2,2,2,1,1,1]$	$(2/9,2/9,2/9,1/9,1/9,1/9)$	**6.102**
$[3,2,2,2,1,1]$	$(3/11,2/11,2/11,2/11,1/11,1/11)$	**4.258**
$[4,3,2,2,1,1]$	$(4/13,3/13,2/13,2/13,1/13,1/13)$	**2.995**
$[3,3,2,2,2,1]$	$(3/13,3/13,2/13,2/13,2/13,1/13)$	**3.201**
$[4,3,3,2,2,1]$	$(4/15,1/5,1/5,2/15,2/15,1/15)$	**2.336**
$[5,4,3,2,2,1]$	$(5/17,4/17,3/17,2/17,2/17,1/17)$	**1.777**

As one can see from the Table 12, in the class of strong weighted majority games the game $[3,2,2,1,1,0]$ provides the minimal variance with value for $nVar = 0.12$. The real decision rule for 1958 is not in the list, because it is not a strong game. However, $nVar$ for this game equals to 0.175, and therefore this rule cannot be optimal even if we accept simple games which are not strong.

Two conclusions can be drawn from this exercise are the following. First, Germany got too little weight as compared to France and Italy. Second, the choice to make Luxembourg a dummy was optimal in our context.

7 Conclusion

In this paper, we have developed a methodology to evaluate and design voting organizations in order to minimize the distance to an egalitarian sharing of a surplus when the process of division across the countries which are members of the organization is described at equilibrium by the nucleolus of the simple game. We have explained why the vector corresponding to the nucleolus can be viewed as a vector measuring the power of each member of the organization when the policy issue has the characteristics of distributive politics.

In the first part of the paper, we have reported our computation results concerning the nucleolus and the Baron-Ferejohn measure of power for the organizations describing the five consecutive stages of the EU. We have also formulated the design issue and alluded to the difficulties attached to the resulting combinatorial problems. For the sake of illustration, we have shown our optimization at works in the case of the EU in 1958. The low size (only 3 players) permits such a brutal approach. Among the lessons of this exercise, we were able to confirm that making Luxembourg a dummy was appropriate but that Germany was mistreated. Certainly the subsequent enlargements of the EU are much more difficult to handle but we plan to address these issues in future research. To do so, we need to handle efficiently all the computational issues attached to the nucleolus (Matsui and Matsui (2000), Wolsey (1976)). Conceivably, to do so, we could consider first to pack the countries into groups of almost equal population and to make two countries from the same group equally desirable in the design of the simple game. Likely, the optimization over such subset of simple games would be more easy to perform.

8 Appendix

8.1 Appendix 1 : Cooperative Games[22], Least Core and Nucleolus

A cooperative game with transferable utility (TU) is a pair (N, V) where $N = \{1, \ldots, n\}$ with $n \geq 2$ is a finite set of players and V is a function that associates a real number $V(S)$ to each subset S of N. It is assumed that $V(\varnothing) = 0$. It is constant-sum if $V(S)+V(N\backslash S) = V(N)$. It is monotonic if : $S \subseteq T \subseteq N \Rightarrow V(S) \leq V(T)$. It is superadditive if $V(S \cup T) \geq V(S)+V(T)$ for all $S, T \subseteq N$ such that $S \cap T = \varnothing$. A player $i \in N$ is a null-player (dummy) of (N, V) if $V(S \cup \{i\}) = V(S)$ $(V(S \cup \{i\}) = V(S) + V\{l\})$. Hereafter, we denote by $X_{PO} \equiv \{y \in \mathbb{R}^n \mid \sum_{i=1}^n y^i = V(N)\}$ the set of (pre)imputations (or Pareto optimal imputations) and by $X_{IR} \equiv \{y \in \mathbb{R}^n \mid \sum_{i=1}^n y^i = V(N),\ y^j \geq V(\{j\}) \forall j \in N\}$ the set of imputations i.e. the set of individually rational preimputations. A player $k \in N$ is at least as *desirable* as a player $l \in N$, denoted $k \succeq l$ if $V(S \cup \{k\}) \geq V(S \cup \{l\})$ for all $S \subseteq N \backslash \{k, l\}$. The *desirability relation* $\succeq$ is reflexive and transitive. If $\succeq$ is complete, the game is called a *complete* game. According to Krohn and Sudhölter (1995), a *directed game* is a complete game such that[23] $1 \succeq 2 \succeq \ldots \succeq n$.

Let X be a compact and convex subset of $\mathbb{R}^n$ and let $x \in X$. We denote by $\theta(x)$ the 2^n-dimensional vector[24] whose components are the numbers $e(S, x) \equiv V(S) - \sum_{i \in S} x^i$ for $\emptyset \subseteq S \subseteq N$ arranged according to their magnitude, i.e., $\theta^i(x) \geq \theta^j(x)$ for $1 \leq i \leq j \leq 2^n$.

[22] See Owen (2001) and Peleg and Sudhölter (2003).

[23] A directed game is the element of the equivalence (with respect to permutations of players) class of complete games where the desirability relation is congruent to the natural order.

[24] This vector is called the vector of excesses attached to x.

The *nucleolus* of (N, V) with respect to X is the unique[25] vector $x^* = N_u(N, V) \in X$ such that $\theta(x^*)$ is minimal, in the sense of the lexicographic order, of the sets $\{\theta(y) \mid y \in X\}$. The nucleolus of (N, V) with respect to X_{IR} will be called hereafter the nucleolus; it is the nucleolus as originally defined by Schmeidler (1969)[26]. We denote also by $\psi(x)$ the 2^{2n}-dimensional vector whose components are the numbers $e(S, x) - e(T, x)$ for $\emptyset \subseteq S, T \subseteq N$ arranged according to their magnitude, i.e., $\psi^i(x) \geq \psi^j(x)$ for $1 \leq i \leq j \leq 2^{2n}$. The *modiclus* is the unique[27] vector $x^{**} \in X_{PO}$ such that $\psi(x^*)$ is minimal, in the sense of the lexicographic order, of the sets $\{\psi(y) \mid y \in X_{PO}\}$.

Given a real number ϵ, the $\epsilon - core$ of (N, V) is the set

$$C_\epsilon \equiv \{x \in X_{PO} : e(S, x) \leq \epsilon \text{ for all } \varnothing \subsetneqq S \subsetneqq N\}.$$

The *least core of* (N, V) denoted $LC(V, N)$[28] is the intersection of all nonempty $\epsilon-$ *core* of (N, V). If (N, V) is superadditive, then $LC(V, N) \subseteq X_{IR}$. In such case, $LC(V, N)$ consists of the vectors x such that $\theta_1(x) = \theta_1(x^*)$. Note that then, $x^* \in LC(V, N)$.

8.2 Appendix 2 : Simple Games[29]

A *simple game* is a pair $(N, \mathcal{W})$ where $N = \{1, \ldots, n\}$ with $n \geq 2$ is a finite set of players and $\mathcal{W}$ is a set of subsets of N satisfying : $N \in \mathcal{W}$, $\varnothing \notin \mathcal{W}$ and $(S \subseteq T \subseteq N$ and $S \in \mathcal{W}) \Rightarrow T \in \mathcal{W}$. The collection $\mathcal{W}$ of coalitions is the set of *winning coalitions*. The simple game $(N, \mathcal{W})$ is *proper* if $S \in \mathcal{W} \Rightarrow N \backslash S \notin \mathcal{W}$. It is *strong* if $S \in \mathcal{W}$ or (and) $N \backslash S \in \mathcal{W}$. It is *constant sum* (self-dual or decisive) if it is proper and strong[30]. Hereafter, we will attach to any simple $(N, \mathcal{W})$ the monotonic TU cooperative game (N, V) where:

$$V(S) = \begin{cases} 1 \text{ if } S \in W \\ 0 \text{ otherwise} \end{cases}$$

Note that (N, V) is superadditive iff $(N, \mathcal{W})$ is proper and that (N, V) is constant-sum iff $(N, \mathcal{W})$ is decisive. A simple game $(N, \mathcal{W})$ is a *weighted majority game* if there exists a vector $\omega = (\omega_1, \ldots, \omega_n; q)$ of $(n+1)$ nonnegative real numbers such that a coalition S is in $\mathcal{W}$ iff $\sum_{i \in S} \omega_i \geq q$; q is referred to as the quota and ω^i is the weight of player $i \in N$. The vector ω is called a *representation* of the simple game $(N, \mathcal{W})$. It is important to note that the same game may admit several representations. A simple game is *homogeneous* if there exists a representation ω such that $\sum_{i \in S} \omega_i = \sum_{i \in T} \omega_i$ for all $S, T \in \mathcal{W}_m$ where $\mathcal{W}_m$ denotes

[25] For a proof of uniqueness, see Peleg and Sudhölter (2003).

[26] In contrast, the *prenucleolus* is the nucleolus with respect to $X \equiv \left\{y \in \mathbb{R}^n \mid \sum_{i=1}^n y^i = V(N)\right\}$. If the cooperative game is zero-monotonic, i.e., if $V(S \cup \{i\}) - V(S) \geq V(\{i\})$ for all $i \in N$ and $S \subseteq N \setminus \{i\}$, the difference between the prenucleolus and the nucleolus vanishes. A simple game is always zero-monotonic unless $\{i\}, S \in \mathcal{W}$ for some $i \in N$ and $S \subseteq N \setminus \{i\}$.

[27] The modiclus has been introduced and studied by Sudhölter (1996, 1997). For a proof of uniqueness, we refer to his original papers or Peleg and Sudhölter (2003).

[28] The notion of least core was first introduced by Maschler, Peleg and Shapley (1979). Each payoff vector of the least core of a zero-monotonic game is individually rational.

[29] See Von Neumann and Morgenstern (1944), Shapley (1962) and Taylor and Zwicker (1999).

[30] Some authors use the term strong for constant sum.

the set of minimal winning coalitions. Such a representation when it exists is referred to as a *homogeneous representation*. We note that if $\omega_i \geq \omega_j$, then player i is at least as desirable as a player j. Finally, we say that a representation is *symmetric*[31] if $\omega_i = \omega_j$ whenever $i \sim j$.

The dual of $(N, \mathcal{W})$ is the simple game $(N, \mathcal{B})$ where $S \in \mathcal{B}$ if and only if $N \backslash S \notin \mathcal{W}$. The collection $\mathcal{B}$ of coalitions is the set of *blocking coalitions*.

8.3 Appendix 3: Representation and Enumeration of Simple Games

A representation of a weighted majority game $(N, \mathcal{W})$ is an *integral representation* if $\omega^i \in \mathbb{N} \cup \{0\}$ for all $i \in N$. Note that, without loss of generality, the quota q can be chosen to be $\underset{S \in \mathcal{W}_m}{Min}\, \omega(S)$. An integral representation ω is *minimal* if there does not exist any integral representation ω' of $(N, \mathcal{W})$ such that $\omega' \leq \omega$. If $\omega \leq \omega'$ for every integral representation ω' of $(N, \mathcal{W})$, then is the *minimum* integral representation of $(N, \mathcal{W})$. A representation is normalized if $q = 1$.

In a strong simple game $(N, \mathcal{W})$, an imputation $x \in X_{IR}$ is a normalized representation of $(N, \mathcal{W})$ if and only if $q(x) \equiv \underset{S \in \mathcal{W}_m}{Min}\, x(S) > \frac{1}{2}$. Peleg (1968) has proved that any imputation in the least core of a strong weighted game $(N, \mathcal{W})$ is a normalized representation of $(N, \mathcal{W})$. Therefore, in particular the nucleolus $x^* ((N, \mathcal{W}))$ of $(N, \mathcal{W})$ is a normalized representation of $(N, \mathcal{W})$. He also proved that if $(N, \mathcal{W})$ is a strong homogeneous weighted majority game, then the nucleolus is the unique normalized homogeneous representation of $(N, \mathcal{W})$ which assigns a zero to each null player. The nucleolus has rational coordinates i.e. can be written as $x^* ((N, \mathcal{W})) = \frac{\omega^*}{\omega^*(N)}$ where the ω^*_i for $i \in N$ are integers whose greatest common divisor is 1. Peleg proves that if $(N, \mathcal{W})$ is strong weighted majority game then ω^* is a minimal integral representation if and only if $\omega^* (N) = 2q (\omega^*) - 1$. He also proved that if if $(N, \mathcal{W})$ is strong homogeneous weighted majority game then ω^* is a minimum integral representation of $(N, \mathcal{W})$. Sudhölter (1996) proved that if $(N, \mathcal{W})$ is a weighted majority game, then the modiclus is a normalized representation of $(N, \mathcal{W})$. Ostman (1987) and Rosenmuller (1987) showed that every homogeneous weighted (not necessarily strong) majority game has a minimal integral representation and that this representation is homogeneous. Sudhölter proved that, up to normalization, this minimal integral representation coincides with the modiclus.

These results point out the existence of relationships between the nucleolus and the set of minimal integral representations. It is important to call the attention on the fact that the combinatorics of these relationships are however quite intricate. Peleg provides an example of a strong weighted simple game with $n = 12$ for which ω^* *is not* a minimal integral representation. Quite remarkably, Isbell (1969) provides an example of a strong weighted simple game with $n = 19$ and a minimum integral representation $\underline{\omega}$ such that: $\omega^* \neq \underline{\omega}$.

Krohn and Sudholter (1995) proved that if $(N, \mathcal{W})$ is a strong weighted majority game and $n \leq 8$, then $LC(N, V) = N_u(N, V)$ which coincides with the unique normalized minimal

[31] We call the attention of the reader on the fact that Freixas, Molinero and Roura () call normalized such representations. We think that this choice of terminology is misleading given the standard use of the word normalized in this area.

integral representation of $(N, \mathcal{W})$. When $n = 9$, they obtain 319124 strong directed games out of which exactly 175428 are weighted majority games. In such a case they show that $LC(N, V)$ is a singleton with the exception of exactly 12 games. Precisely, all strong weighted majority games with $n = 9$ have a unique minimal normalized representation which coincides with the least core and thus with the nucleolus with the exception of 14 games which have exactly two minimal representations differing on one type of players. Moreover, in 12 of these games both representations are exactly the extreme points of the least core. In the remaining two cases, no normalized representation is contained in the least core though the set is a singleton (i.e. coincides with the nucleolus)[32]. Freixas and Molinero (2009) also prove that when $n = 9$ any strong weighted majority game admits a unique minimal normalized symmetric representation but when $n = 10$, there are strong weighted majority games without a unique minimal symmetric representation and with more than two minimal integral representations.

The enumeration of all simple games or important subclasses like for instance the subclasses of strong, complete, directed, weighted majority or subclasses obtained by intersection of these subclasses is important for the combinatorial optimization conducted in our paper[33]. This paper has been a topic of investigation since von Neumann and Morgenstern who enumerated all strong simple games when $n = 5$ and Gurk and Isbell (1959) who enumerated all strong simple games when $n = 6$. Isbell (1959) provides the list of the 135 strong weighted majority games when $n \leq 7$ together with their unique minimal integral representations; 38 of those games are homogeneous. Table 13 below reproduces the enumeration derived by Krohn and Sudhölter for games[34] with $n \leq 9$.

Table 13:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
# directed games	3	5	10	27	119	1173	44315	161175190	?
# weighted majority games	3	5	10	27	119	1113	29375	2730166	?
# strong directed games	1	1	2	3	7	21	135	2470	319124
# strong weighted majority games	1	1	2	3	7	21	135	2470	175428
# homogeneous games	1	3	8	23	76	293	1307	6642	37882

The enumeration of all simple games (including the two constant ones attached to $V(\emptyset) = 1$ and $V(N) = 0$) is known as the Dedekind's problem. Table 14 below reproduces the enumeration for games with $n \leq 6$.

The enumeration of all strong simple games[35] (including the two constant ones corresponding to $V(\emptyset) = 1$ and $V(N) = 0$) has also attracted attention. Table 15 below,

[32] This result was also proved by Freixas, Molinaro and Roura (2007). They also prove that in the case where $n \leq 7$, all weighted majority games have a unique minimal integral representation. Finally, they prove that when $n = 8$, they are 154 weighted majority games with two minimal integral representations (of course, we know from above that none of them is strong). They show however than they all have a unique minimal symmetric integral representation.

[33] We may also consider those satisfying some symmetry conditions as in Loeb and Conway (2000).

[34] In this enumeration, they dont assume that $\emptyset \notin \mathcal{W}$, $N \in \mathcal{W}$.

[35] Strong simple games are also often called maximal intersecting families of sets.

Table 14:

n	1	2	3	4	5	6
# simple games	3	6	20	168	7581	7828354

extracted from Loeb and Conway (2000), reproduces the enumeration for games with $n \leq 8$.

Table 15:

n	1	2	3	4	5	6	7	8
# strong simple games	1	2	4	12	81	2646	1422564	229809982112

True, the enumerations in tables 14 and 15 count games which are isomorphic. If not, the numbers decrease in a significant way as illustrated in table 16 below for games with $n \leq 7$.

Table 16:

n	1	2	3	4	5	6	7
# isomorphism classes of strong simple games	1	1	2	3	7	30	716

As already pointed out, we may also want to enumerate simple games satisfying some symmetry properties described through the group of permutation automorphisms preserving the set of minimal winning coalitions. Along these lines, we may also limit the enumeration to games where some players are always treated similarly (the set of players is partitioned into a number of types where two players from the same type are perfect substitutes in the simple game). Freixas and Molinero (2009) and Kurz and Tautenhahn (2010) have derived formulas to enumerate all such simple games.

References

[1] Algaba, E. Bilbao, J.M and J.R. Fernandez (2007) "The Distribution of Power in the European Constitution", *European Journal of Operational Research,* 176, 1752-1766.

[2] Alon, N. Edelman, P.H. (2010) "The Inverse Banzhaf Problem", *Social Choice and Welfare,* 34, 371-377.

[3] Ansolabehere, S. Gerber, A. and J. Snyder (2002) "Equal Votes, equal Money: Court-Ordered Redistricting and Public Expenditures in the American States", *American Political Science Review,* 96, 767-777.

[4] Banks, J.S. (2000) "Buying Supermajorities in Finite Legislatures", *American Political Science Review,* 94, 677-681.

[5] Banks, J.S. and J. Duggan (2000) "A Bargaining Model of Collective Choice", *American Political Science Review,* 94, 73-88.

[6] Banzhaf, J.F. III. (1965) "Weighted Voting Doesn't Work : A Mathematical Analysis", *Rutgers Law Review,* 19, 317-343.

[7] Banzhaf, J.F. III. (1968) "One Man 3.312 Votes: A Mathematical Analysis of the Electoral College", *Villanova Law Review*, 13, 304-332.

[8] Barbera, S. and M.O. Jackson (2006) "On the Weights of Nations : Assigning Voting Weights in a Heterogeneous Union", *Journal of Political Economy*, 114, 317-339.

[9] Baron, D. P. and J. A. Ferejohn (1989) "Bargaining in Legislatures", *American Political Science Review,* 83, 1181-1206.

[10] Barr, J. and F. Passarelli (2009) "Who Has the Power in the EU?", *Mathematical Social Sciences,* 57, 339-366.

[11] Beisbart, C., Bovens, L. and S. Hartmann (2005) "A Utilitarian Assessment of Alternative Decision Rules in the Council of Ministers", *European Union Politics*, 6, 395-418.

[12] Beisbart, C. and S. Hartmann (2010) "Welfarist Evaluations of Decision Rules under Interstate Utility Dependencies", *Social Choice and Welfare*, 34, 315-344.

[13] Bilbao, J.M., Fernandez, J.R., Jimenez, J.J. and J.J. Lopez (2002) "Voting Power in the European Union", *European Journal of Operational Research,* 143, 181-196.

[14] Diermeier, D. and R.B. Myerson. (1999) "Bicameralism and its Consequences for the Internal Organization of Legislatures". *American Economic Review,* 89, 1182-1196.

[15] Eraslan, H. (2002) "Uniqueness of Stationary Equilibrium Payoffs in the Baron-Ferejohn Model", *Journal of Economic Theory,* 103, 11-30.

[16] Eraslan, H., A. McLennan (2006) "Uniqueness of Stationary Equilibrium Payoffs in Coalitional Bargaining", Mimeo.

[17] Felsenthal, D. S. and M. Machover (1998) *The Measurement of Voting Power. Theory and Practice, Problems and Paradoxes*, Cheltenham: Edward Elgar.

[18] Felsenthal, D. S. and M. Machover (2001) "Myths and Meanings of Voting Power: Comments on a Symposium", *Journal of Theoretical Politics* 13(1), 81-97.

[19] Felsenthal, D. S. and M. Machover, 2004. "Analysis of QM rules in the draft constitution for Europe proposed by the European Convention, 2003", *Social Choice and Welfare* 23, 1-25.

[20] Freixas, J., Molinaro, X. and S. Roura (2007) "Minimal Representations for Majority Games", in *Computations and Logic in the Real World*, Lecture Notes in Computer Science, Vol 4497, Springer Verlag, 297-306.

[21] Freixas, J. and X. Molinaro (2009) "On the Existence of Minimum Integer Representation for Weighted Voting Systems", *Annals of Operations Research*, 166, 243-260.

[22] Freixas, J. and X. Molinaro (2009) "A Fibonnaci Sequence for Linear Structure with Two Types of Components", Mimeo.

[23] Garrett, G. and G. Tsebelis (1999) "Why Resist the Temptation to Apply Power Indices to the European Union", *Journal of Theoretical Politics*, 11, 291-308.

[24] Garrett, G. and G. Tsebelis (2001) "Even More Reasons to Resist the Temptation of Power Indices in the European Union", *Journal of Theoretical Politics*, 13, 99-105.

[25] Groseclose, T. and J.M. Snyder. (1996) "Buying Supermajorities", *American Political Science Review,* 90, 303-315.

[26] Gurk, H.M. and J.R. Isbell (1959) "Simple Solutions" in *Annals of Mathematics Studies, Contributions to the Theory of Games IV*, Volume 40, Princeton University Press, 247-265.

[27] Horiuchi, Y. and J. Saito (2003) "Reapportionment and Redistribution: Consequences of Electoral reform in Japan", *American Journal of Political Science*, 47, 669-682.

[28] Isbell, J.R. 1956. "A Class of Majority Games", *The Quarterly Journal of Mathematics*, 7, 183-187.

[29] Isbell, J.R. 1959. "A Counterexample in Weighted Majority Games", *Proceedings of the American Mathematical Society*, 20, 590-592.

[30] Isbell, J.R. (1959) "On the Enumeration of Majority Games", *Mathematical Tables and Other Aids to Computation,* 13, 21-28.

[31] Isbell, J.R. (1969) "A Counterexample in Weighted Majority Games", *Proceedings of the American mathematical Society,* 20, 590-592.

[32] Krohn, I. and P. Sudholter (1995) "Directed and Weighted Majority Games", *Mathematical Methods of Operations Research*, 42, 189-216.

[33] Kurz, S. and Tautenhahn (2010) "On Dedekind's Problem for Complete Simple Games", Mimeo.

[34] Laruelle, A. and F. Valenciano (2008) *Voting and Collective Decision Making*, Cambridge University Press.

[35] Laruelle, A. and M. Widgren (1998) " Is the Allocation of Voting Power among the EU States Fair", *Public Choice*, 94, 317-339.

[36] Le Breton, M. and V. Zaporozhets (2010) "Sequential Legislative Lobbying under Political Certainty", *Economic Journal*, 120(543), 281-312.

[37] Le Breton, M., Sudhölter, P. and V. Zaporozhets (2010) "Sequential Legislative Lobbying", DP No. 8/2009, Department of Business and Economics, University of Southern Denmark.

[38] Laruelle, A. and M. Windgren (1998) "Is the Allocation of Voting Power Among EU States Fair?", *Public Choice,* 94, 317-339.

[39] Leech, D. (2002) "Designing the Voting System for the Council of Ministers of the European Union", *Public Choice*, 113, 437-464.

[40] Loeb, D.E. and A.R. Conway (2000) "Voting fairly: Transitive Maximal Intersecting Family of Sets", *Journal of Combinatorial Theory, Series A*, 91, 386-410

[41] Matsui, T. and Y. Matsui (2000) "A Survey of Algorithms for Calculating Power Indices of Weighted Majority Games", *Journal of the Operations Research,* 43, 71-85.

[42] Maaser, N. and S. Napel (2007) "Equal Representation in Two-Tier Voting Systems", *Social Choice and Welfare*, 28, 401-420.

[43] Megiddo, N. (1971) "The Kernel and the Nucleolus of a Product of Simple Games", *Israel Journal of Mathematics,* 9, 210-221.

[44] Megiddo, N. (1974) "Nucleoluses of Compound Simple Games", *SIAM Journal of Applied Mathematics,* 26, 607-621.

[45] Montero, M. (2005) "On the Nucleolus as a Power Index", *Homo Oeconomicus,* 4, 551-567.

[46] Montero, M. (2006) "Noncooperative Foundations of the Nucleolus in Majority Games", *Games and Economic Behavior,* 54, 380-397.

[47] Montero, M. (2007) "The Paradox of New Members in the Council of Ministers: A Noncooperative Approach", *CeDEx Discussion Paper,* No 2007-12.

[48] Napel, S. and M. Widgrén (2004) "Power measurement as Sensitivity Analysis", *Journal of Theoretical Politics*, 16, 517-538.

[49] Napel, S. and M. Widgrén (2006) "The Inter-Institutional Distribution of Power in EU Codecision", *Social Choice and Welfare*, 27, 129-154.

[50] Napel, S. and M. Widgrén (2009) "Strategic versus Non-Strategic Voting Power in the EU Council of Ministers: The Consultation Procedure", Mimeo.

[51] O'Neill, B. and B. Peleg. 2000. "Reconciling Power and Equality in International Organizations: A Voting Method from Rabi Krochmal of Kremseir", *Jewish Political Studies Review*, 12, 67-81

[52] Ostmann, A. (1987) "On the Minimal Representation of Homogeneous Games", *International Journal of Game Theory*, 16, 69-81.

[53] Owen, G., 2001. *Game Theory*, California: Academic Press.*Cooperative Games*, Kluwer Academic Publishers.

[54] Passarelli, F. and J. Barr (2007) "Preferences, the Agenda Setter, and the Distribution of Power in the EU", *Social Choice and Welfare,* 28, 41-60.

[55] Peleg, B. (1968) “On Weights of Constant-Sum Majority Games”, *SIAM Journal on Applied Mathematics*, 16, 527-532.

[56] Peleg, B. and P. Sudhölter (2003) *Introduction to the Theory of Cooperative Games*, Boston, Kluwer Academic Publishers.

[57] Penrose, L. S. (1946) "The elementary Statistics of Majority Voting", *Journal of the Royal Statistical Society,* 109, 53-57.

[58] Rosenmüller, J. (1987) "Homogeneous Games : Recursive Structure and Computations", *Mathematics of Operations Research*, 12, 309-330.

[59] Schmeidler, D. 1669. “The Nucleolus of a Characteristic Function Game”, *SIAM Journal on Applied Mathematics*, 17, 1163-1170.

[60] Shapley, L.S. 1962. “Simple Games: An Outline of the Descriptive Theory”, *Behavioral Sciences*, 7, 59-66.

[61] Shapley, L.S. and M. Shubik (1954) ”A Method for Evaluating the Distribution of Power in a Committee System”, *American Political Science Review*, 48, 787-792.

[62] Snyder, J.M., Ting, M. M. and S. Ansolabehere (2005) ”Legislative Bargaining under Weighted Voting”, *American Economic Review,* 95, 981-1004.

[63] Steunenberg, B., Schmidtchen, D. and C. Koboldt (1999) "Strategic Power in the European Union : Evaluating the Distribution of Power in Policy Games", *Journal of Theoretical Politics*, 11, 339-366.

[64] Taylor, A.D. and W.S. Zwicker (1999) *Simple Games*, Princeton University Press.

[65] Tsebelis, G. (1994) "The Power of the European Parliament as a Conditional Agenda Setter", *American Political Science Review*, 88, 128-142.

[66] Von Neumann, J. and O. Morgenstern. (1944) *Theory of Games and Economic Behavior*, Princeton: Princeton University Press.

[67] Wolsey, L.A. (1976) "The Nucleolus and Kernel of Simple games or Special valid Inequalities for 0-1 Linear Integer programs", *International Journal of Game Theory*, 5, 225-238.

[68] Young, H.P. (1978a) “A Tactical Lobbying Game” in *Game Theory and Political Science*, Ordeshook, P.C. (Ed), New York University Press, New York.

[69] Young, H.P. (1978b) “The Allocation of Funds in Lobbying and Campaigning”, *Behavioral Science*, 23, 21-31.

[70] Young, H.P. (1978c) "Power, Prices, and Incomes in Voting Systems", *Mathematical Programming*, 14, 129-148.