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NETWORKS, REPLICATOR DYNAMICS AND THE PROPAGATION OF IDIOSYNCRATIC SHOCKS

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[Very first draft][1]

[1] The paper is currently being copy-edited but due to deadlines we must submit it without these corrections. However, it will be copy-edited before the article is posted on the congress website.

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Abstract

This paper develops a unified framework for the study of how intra-network relation can appears as propagation mechanism of idiosyncratic shocks. This network leads to a reduced replicator dynamic model with heterogeneous firms and particular evolutionary game that we call *idiosyncratic game*. Under the assumption that idiosyncratic shocks are followed by strategy change in firm, we provide a fairly characterization of the condition of propagation of idiosyncratic game, underlying the importance of network structure and intra-network relation as interaction relation and power relation. We show these types of *idiosyncratic game can be* characterized by a perfect information cooperation game, oriented by the firm with better power degree under certain condition, and almost always resulting in the least bad possible way for the latter.

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1 Introduction

Recent works on the origins of aggregate fluctuations highlight the importance of microeconomics shock in the volatility of macroeconomic aggregate. These works rekindled interest in complex economic or social interlinkages as channels for propagation and amplification of shocks. The idea that firm's interlinkages could lead to the propagation of idiosyncratic shock is now well established, and a lot of studies have focused on it.

Acemoglu et al. (2012, 2014) argued that idiosyncratic shock at firm level can propagate over input-output linkages within economy, with potentially significant consequences for macroeconomic volatility and economic growth. On other hand, Barrot and Sauvagnat (2016) underlined the importance of input-specificity between firms in the propagation process of such shocks. It seems that the economic structure and the particular linkages between firms composing it are fundamentals in the propagation of such shocks. Much of work on the subject give important insight on the economic condition under which idiosyncratic shock could propagate, as important interlinkages, input specificity or fat tailed firm size distribution (Gabaix, 2011), but for the moment, these works go in many different directions and do not provide clear indications on the way that such condition influence the propagations of idiosyncratic shock.

Our aim is to unify and improve the understanding of the principal mechanism at work in the propagation of idiosyncratic shocks and to provide answer to the following question: why does idiosyncratic shock propagate?

For this purpose, we need to establish how network work and permit propagation. We based our work on evolutionary game theory and in particular on replication dynamic. We start with a reduced form model of network composed by two heterogenous firms, a *hub-firm*[2] and a *partner firm*. The model relies on a general framework describing the structure of the network and the relation between firms constituting it. This framework is built around three major assumptions: i) intra-network relations are composed by interaction relation described by an interaction function and asymmetrical power relation (Chassagnon, 2019); ii) firm hit by an idiosyncratic shock take adaptive action which can be seen as strategy switch; iii) strategy implemented by *hub-firm* act as a generative replicator, which mean that she had the opportunity the propagate trough the network via imitation mechanism.

[2] The specific role of the *hub-firm* is described in section 2.

We first show that our framework opens the way to understand what we call *idiosyncratic games*. An *idiosyncratic game* corresponds to the specific game where the *hub firm* is hit by an idiosyncratic shock (positive or negative). We show that idiosyncratic shock can lead to a variety of different *idiosyncratic games*, and how in each game idiosyncratic shock eventually spreads to the *partner firm*. These types of game highlight that the propagation of initial idiosyncratic shock on *hub-firm* correspond to the best situation for this latter. In fact, the adaptative strategy from *hub-firm* and so the idiosyncratic shock is internalized voluntarily or under pression of the *hub-firm*, by the *partner firm*. Three different cases are observable, in the first one the adaptative strategy associated to the idiosyncratic shock correspond to a Nash equilibrium leading to the propagation of shock. In the second case this adaptative strategy does not correspond to a Nash equilibrium, and the propagation of shock relies on the interaction between firm. Finally, if the interaction between firms is not high enough to spread adaptative strategy, the propagation relies on exercise power by *hub-firm* on *partner firm.*

We illustrate these results by simulation of these three types of *idiosyncratic game* and show that the propagation of initial idiosyncratic shock relies on three non-additive condition: i) the payoff distribution of adaptative strategy; ii) the level of interaction between firms; iii) the degree of power exercise by the *hub-firm* on *partner firm.*

These results highlight that relying on replicator dynamic, *idiosyncratic games* can be characterized by a perfect information cooperation game, that can be oriented by the *hub-firm* under certain condition, and almost always resulting in the least bad possible way for the *hub-firm.*

Related Literature As already indicated, this paper relates to several strands of literature such as the literature on the idiosyncratic origins of aggregate fluctuations, on firms' network, network game, and evolutionary game theory.

The critical building block of our general framework is the idiosyncratic origins of aggregate fluctuations. As such, our paper relies on the seminal work of Long and Plosser (1983) and on the various recent theoretical and empirical contribution on the idiosyncratic origins of aggregate fluctuations such as Gabaix (2011), Acemoglu et al. (2012, 2014), di Giovani et al. (2014), Barrot and Sauvagnat (2016), Baqaee and Farhi (2018, 2019), Baqaee and Rubbo (2022). These works give the basic to understand the microeconomic origins of aggregates fluctuations, and gives some insight on the necessary condition to the propagation of such shocks. We build our framework on these particular conditions, trying to unify it trough another spectrum to study the propagation of idiosyncratic shock.

This paper is also closely related to the evolutionary game theory, and in particular replication dynamic. The paper of Neuman (1966) and the seminal works of Maynard Smith (1973, 1982, 1988) on the analysis on competition as strategy open that opened the way to the work on Generalized Darwinism (GD) presented as a possible unifying framework for the evolutionary approaches developed in economics and in management and organization studies by Hodgson and Knudsen (2003, 2007, 2010), Johansson and Kask (2013), Hodgson and Stoelhorst (2014), Chassagnon and Brette (2021). Even though the literature on evolutionary game theory does not generally consider the propagation of idiosyncratic shock, our results highlight that the evolutionary game theory could be useful in describing the firm comportment in face of these types of shock and the consequences on their propagation.

This paper is also closely related to the literature on the firm network and environment and particularly on the seminal work of Chassagnon (2014, 2019) on the theory of the firm as power-based entity (TFPBE) or on the papers of Jacobides et al. (2018) and Acemoglu et al. (2016).

Finally, the paper also relies on literature on network game and builds on various different contribution on the network game literature, such as Acemoglu et al. (2015), Calvó-Armengol and Zenou (2004), Ballester et al. (2006), Candogan et al. (2012), Allouch (2012), Badev (2013), Bramoullé et al. (2014) and Elliott and Golub (2014).

Outline The rest of this paper is organized as follow. In section 2, we provide our general framework, laying done the main assumptions of the paper. In section 3, we provide replication dynamic model based on imitation at the origin of the propagation of idiosyncratic shocks. Section 4 introduces *idiosyncratic game* and presents further evidence on the resolution of these games. Section 5 provides simulation of different *idiosyncratic games*, corroborating the main results obtains in past section. Section 6 concludes.

2 General framework

This section aims to describe why and how idiosyncratic shock could propagate trough firm's network. Our general framework is characterized by three assumptions on the environment, the characterization of idiosyncratic shocks and the transmission process of idiosyncratic shocks. These three assumptions rely on two fundamental theoretical characterization, knowing the theory of the firm as power-based entity (TFPE) and the replicator dynamic.

2.1 Microeconomic foundations

Our general framework is designed to respond to one of the main pitfalls of works on the idiosyncratic origins of aggregates fluctuations. The primary objective of these models was to demonstrate the idiosyncratic origin of aggregate fluctuations, the difficulty of such an approach lies in the fact that the idiosyncratic origin of aggregate fluctuations remains for the moment a not well understood phenomenon.

One of the most important pitfalls of these works is that they provide evidences on the idiosyncratic origins of aggregate fluctuations and on the necessary conditions for is existence, but they do not provide a real explanation on the reason of this origin. Works of Gabaix (2011), Acemoglu et al. (2012) or Barrot and Sauvagnat (2016) demonstrate that idiosyncratic shock could be the origins of aggregate fluctuations when there is a fat-tailed distribution of firms, intersectoral input–output linkages, or input specificity, but they do not provide evidence on the way that this type of shock could propagate. We argue that to go further on this topic, more microeconomics foundations are required.

Since this new research field lay on firms' behavior, it is necessary to know more about it. The propagation process is essentially based on the adjustment of prices and quantities[3] following an idiosyncratic shock. This vision of the propagation is based on the idea that the relations constituting the production network are only simple market relations. In reality, a part of the economic relationship is based on an institutionalized inter-firm cooperation that goes far beyond the simple commercial relationship. Chassagnon (2019) speaks of vertical quasi-integration, characterizing a form of organization of production that aims to take advantage of market benefits while promoting a market logic. This type of subcontracting is characterized by a cooperative organizational structure bringing together the different firms of the network through contractualization methods of relational type. Therefore, considering that the propagation of idiosyncratic shocks is not only achieved through price and quantity mechanisms becomes quite possible. The question is then to know what the other mechanisms at work during the propagation of idiosyncratic shocks are if the intra-network relations are not only governed by the market relation. The current vision of the relations constituting the production network are for the moment insufficient to allow a full understanding of these phenomena, we need a framework where we could understand better the firm's behaviors, relations and decisions.

[3] See Grassi (2017) or Baqaee and Rubbo (2018).

Consequently, the next three subsection aim to lay the microeconomic foundations in the understanding of idiosyncratic shock propagation based on the TFPBE (Theory of the Firm as Power-Based Entity) and replicator dynamics.

2.2 Environment

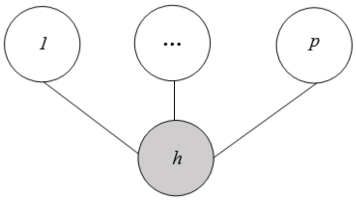
The first aspect of our general framework we need to address is the environment. In order to characterize the environment in which we place ourselves to understand the propagation of idiosyncratic shocks, we will rely on the theory of the firm as a power-based entity (Chassagnon, 2019). Different aspect of this theory will be used to describe the environment.

The theory of the firm as power-based entity deals with firm's production network and the intra-network relations. The production process has evolved and is now based on inter-firms cooperations. This cooperation can be seen as recurring subcontracting relationships between different and independent firms at different level of the production process. These relationships exist trough a demand of specific production formalized and rationalized by the client firm. Chassagnon (2019) explained that this evolution led to a new organizational production called the network-firm.

The organizational architecture of the network is pyramidal, i.e., that the inter-firm work fragmentation is composed by at least two level of hierarchy with different degree of responsibility. The first one concern the *hub-firm* firm (usually superstar firms), responsible of the organization of the network, and the second one concern *partner firms* involved in the production process. The *hub-firm* administrates responsibility to *partner firms*, but keep in way some control on them by influencing the *partners firm's* choices on others levels. The governance of the network-firm links autonomous firms that produce modules requiring joint action. The member firms participate in a process of de facto vertical integration based on mutual interdependence. Based on this characterization of network firm, we can establish our first assumption.

Definition 1 (Network configuration). *For a network firm $\mathcal{N}$ composed by one hub-firm h and $P = \{1, \dots, p\}$ a finite set of partner firms, the network can be represented by an undirected graph as follow:*

Figure1: Network structure



Where each vertex corresponds to a firm in the network and each edge correspond to the relations between firms.

In order to define the environment properly, we now need to address the intra-network relation nature. We can distinguish two different type au relation inside the network. First of all, the network described as in the TFPBE, is an interaction network. So, the first type of relation inside the network can be defined as interaction relation. Basically, we can establish that different state firm inside the network are interlinked. We can find such consideration in Acemoglu et al. (2015). Interdependencies between different intra-network firm, may arise due to strategic consideration, contractual agreements, or some exogenous constraints on the agents.

The second type of intra-network relation that we can observe are power relation. These type of relation remains on *de facto* power and are asymmetrical. Power can be seen as the latent capacity that an entity *A* have to constrain and shape the choices of an entity *B* in such way that the behaviors of entity *B* are oriented in direction favorable to entity *A*.[4] This mean that for each interacting peer of firms, each firm exercise power influence on other but with different degree. The power can be exercise or not.

Assumption 1 (Relation function) *For a network firm $\mathcal{N}$ composed by one hub-firm h and $P = \{1, \dots, p\}$ a finite set of partner firms, the state of any given firm i depend on the states of other firms and on the power influence of the hub-firm via the relationship*

$$x_i = f\left(\sum_{j=1}^{k} w_{ih} x_h\right) + \beta . e^{\left(\sum_{j=1} k \rho_{hi}\right)} \tag{1}$$

Where x_i and x_h are respectively the state of firm i and of the hub-firm, f is continuous and increasing function which refer to as the economy's interaction function and w_{ih} is a constant

[4] See Sterelny et al. (1996, p. 395).

capturing the extent of interaction between firm i and h, ρ_{hi} *is a constant capturing the power influence of firm h on firm i, k is a constant and* β *is binary variable traducing the exercise of power or not.*

Given assumption 1 we can now redefine the network structure by introducing relation.

Definition 2 (Network structure with relation) *For a network firm* $\mathcal{N}$ *composed by one hub-firm h and* $P = \{1, \dots, p\}$ *a finite set of partner firms and by interaction relation and power relation, the network can be represented by a directed graphs as follow where edge correspond to the relations between firms and bold edges represent asymmetrical power.*

Figure 2 : Interaction relation

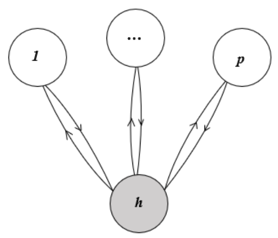
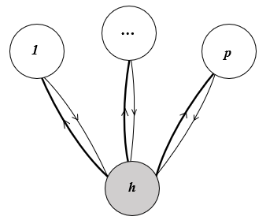


Figure 3 : Power relation



2.3 Idiosyncratic shock and strategy

Another aspect that is important to clarify is the one of the shocks. We try here to give a definition to what we call an idiosyncratic shock and how we can interpret it in relation to the firm. That is a fundamental point to understand the transmission process we will treat after that. We can define an idiosyncratic shock as a shock to a firm's that is not caused by macroeconomic fluctuations or sectoral shocks that is a change in, for example, output that does not simultaneously affect all firms in the economy or the entire sector. The traditional view of these type of shocks suggests that their occurrence results in some change in the firm characteristics.[5] For example, much of the works on this topic base their idiosyncratic shock identification on change in, for example, sales or output in the firms, as if a shock occurred and change in these characteristics happen mechanically.

We think at this stage that it is necessary to detailed a little more the process of idiosyncratic shocks. In our view, an idiosyncratic shock is composed by at least two elements. i) a stimulus that will be received by a firm and ii) the fallout (positive or negative) on the firm induced by the stimulus. The fallout is composed by at least an adaptative action from the firm, and in some cases by uncontrolled consequences of the stimulus. This opens the way to two possible

[5] See Gabaix (2011) or di Giovani et al. (2014) for some examples.

cases. In the first one, a stimulus happens and the firm act adaptively to prevent (enhance) the uncontrolled consequences of the stimulus. In the second case, the stimulus happens giving uncontrolled consequences and the firm act adaptively to counterbalance (enhance) these uncontrolled consequences. So, from this point of view, an idiosyncratic shock is always accompanied by an adaptative action from the firm, to prevent or to compensate (enhance) the uncontrolled repercussion of the shock. Based on this, we obtain the following definition of idiosyncratic shock.

Definition 3 (Idiosyncratic shock) *for a given firm i, an idiosyncratic shock ε_i is composed by a stimulus σ_i, an adaptative response from the firm α_i to the stimulus and uncontrolled repercussion on firm characteristics ϵ_i:*

$$\varepsilon_i = \sigma_i + \underbrace{\alpha_i + \epsilon_i}_{\text{fallout}} \tag{2}$$

Let illustrate it with two examples: let first imagine a firm facing a drop of the demand, then sales will drop too, in this case the drop of demand correspond to the stimulus and the drop of sales correspond to the uncontrolled consequences. But, in order to compensate the drop of sales, firm will act on some other aspect like on total output (that is the adaptative action of the firm). We can also imagine a firm facing an aleatory stimulus (like a drop in input), for the moment the stimulus does not exercise any uncontrolled consequences, so the firm decide to act to prevent the uncontrolled cause, for example, by reducing the expected output. In this case, the firm deliberately internalize the shock in order to control it, meaning that there are no uncontrolled consequences.

In traditional view of the idiosyncratic shocks, works focuses on the uncontrolled part of the shock. In this paper we will focus on the adaptative action from the firm, because we think it's a relevant part of idiosyncratic shocks that which always exist in the case of idiosyncratic shock. This an important definition in identification of idiosyncratic shock, at least theoretically.

Since firm's always take action when facing idiosyncratic shocks, in order to prevent or compensate (enhance), we can now propose a new assumption that is that a striking shock on firm is followed by an adaptative action that can be seen as a change in set of strategy.

Assumption 2 (firm strategy) *for a given firm i following a strategy x, the occurrence of an aleatory idiosyncratic shock* ε_i *will be accompanied by a firm switch from strategy x to strategy y that is more adapted.*

$$x_i + \varepsilon_i = y_i \tag{3}$$

Given definition 3 and assumption 2, the occurrence of shock means a larger impact that usually admitted. In fact, an idiosyncratic shock materialized by a stimulus could affect a set of different strategy when originally shock on one aspect had only linked impact. This mean that a change in the set of strategy can be due to a large type of shocks.

2.4 Transmission

Now that the structure of a network and the relation between idiosyncratic shock and strategy have been defined, we can move on the transmission process of shock. We make the central assumption that this propagation process relies on the concept of replicator. A replicator is an evolutionary game theory concept and can be seen as a copy of an element carrying information about this element in virtue of being relevantly similar to it, it is obtained through a process of replication and where the element copied plays a causal role in the production of the copy, finally the copy has the same, or similar, functional capacities of the element copied. Hodgson and Knudsen (2003, 2007, 2010) used this definition and proposed that the firm could be seen as an interactor within which replication process operates and Chassagnon and Brette (2021) proposed that Business model could be seen as replicator.

We move on these propositions on propose that the set of strategy set up by firm can be seen as replicator and so a process of replication based on imitation is possible allowing us to appreciate the evolution of different strategies in the populations of firms composing the network. In fact, is it possible to appreciate the replication dynamic of strategies. In this framework, firms are programmed to pure strategy and review their strategy, sometimes resulting in strategy change. There are two basic elements. The first one is a specification of time rate $r_i(x)$ at which firms review their strategy choice for a firm who use pure strategy i from a set of k strategy. The second one is a specification of the choice probabilities of a reviewing firm. The probability P_i^j that a reviewing i-strategist firm will switch to some pure strategy j may here depend on the current performance of these strategy and other aspects of the current population state.

Assumption 3 (Replication dynamic) *for a finite population of firm programmed in pure strategy, the replication dynamic $\dot{N}_i$ of strategy i, given initial population state N_i programmed in pure strategy i and N_j programmed to pure strategy j*

$$\dot{N}_i = \sum_{j\in k} N_j r_j(N) P_i^j(N) - N_i r_i(N) P_j^i(N) \quad (4)$$

Where the first term corresponds to the inflow from subpopulation j and the second term correspond to the outflow from subpopulation i

If we maintain assumption 2, we can state that an idiosyncratic shock is accompanied by a change of strategy in *hub-firm*, and observe how this strategy will evolve in the network and so the shock.

3 Model

In this section, we sketch the replicator dynamic model explaining the propagation of idiosyncratic shocks trough network. We start from particular condition of network structure and idiosyncratic shock and we show how they propagate trough network.

We consider a network with a population *N* composed by two firms, a *hub-firm* $h \in N$, and a *partner firm* $p \in N$. Based on the assumption 3, we assume that initially the two firms play the same strategy *x* (with idiosyncratic differences and different payoffs). We suppose now, that an idiosyncratic shock occurs on *h*, leading the firm *h* to modify its strategy toward strategy *y* that is more adapted taking into account the idiosyncratic shock. We now focus on the behavior of firm *p* knowing that firm *h* is programmed in strategy *y*.

3.1 Utilities

Utility refers to the reward (or cost) obtained by firms interacting with other firms inside the network. Let *S* be the finite set of pure strategies of firm $i \in N$ and *T* the finite set of pure strategies of firm $j \in N$. For any strategy profile $s \in S$, and firm $i \in N$, and any strategy profile $t \in T$ of and firm $j \in N$, let $s_t(i) \in R$ be the associated payoff to player *i* and $t_s(j)$ the associated payoff to player *j*.[6]

In our case, after the occurrence of idiosyncratic shock on firm *h*, firm *p* has choice between staying programmed in strategy *x* and switch toward strategy *y*. So, weed need to establish the two conditional utilities of these strategies.

[6] If there are two players *I* and *j* and two strategies *x* and *y*, $x_y(i)$ designed the payoff of strategy *x* played by player *I* meanwhile player is programmed in strategy *y*.

The utility of firm *p* for strategy *x* given that firm *h* is programmed in strategy *y* is given by:

$$u_x(p) = x_y(p)^3 + x_y(p).f\left(\sum_{y=1} w_{hp}.y_x(h)\right) + \beta_1.e^{\left(\sum_{y=1} k\rho_{hp}\right)} \quad (5)$$

And utility of firm *p* for strategy *y* given that firm *h* is programmed in strategy *x* is:

$$u_y(p) = y_y(p)^3 + y_y(p).f\left(\sum_{y=1} w_{hp}.y_y(h)\right) + \beta_2.e^{\left(\sum_{y=1} k\rho_{hp}\right)} \quad (6)$$

Where $x_y(p)$ and $y_y(p)$ are respectively the payoff obtained by firm *p* when strategy *x* and strategy *y* are played meanwhile firm *h* play strategy *y*. $y_x(h)$ is the payoff obtained by firm *h* when strategy *y* is played meanwhile firm *p* plays strategy *x* and $y_y(h)$ the payoff obtained by firm *h* when strategy *y* is played meanwhile firm *p* plays strategy *y*. f is a continuous and increasing function[7], the constant $w_{hp} \geq 0$ captures the extent of interaction between firm *h* and firm *p*. Higher w_{hp} means that the utility of firm *p* is more sensitive to the payoff of firm *h*. So, we can call $f\left(\sum_{y=1} w_{m,p}.y_y(h)\right)$ the interaction function.[8]

$\rho_{hp} \geq 0$ is a constant that capture the power influence exercise by firm *h* on firm *p*[9], *k* is a constant that permit to keep ρ between 0 and 1, and β_1 and β_2 are a binary variables. If $\beta = 0$, firm *h* doesn't exercise his power on firm *p* and if $\beta = 1$ firm h exercises his power on firm *p*. This mean that when firm *h* exercises his power on firm *p*, utility of firm *p* depends on his payoff, on the interaction with other firms, and on the influence exercises by firm *h* on firm *p*.

3.2 Evolutionary process

The evolutionary process (mechanism by which firms reproduce their strategy in the population) is based on one fundamental mechanism which is "imitation". The imitation mechanism makes firms tend to imitate the strategies that provide higher utilities within the network. We consider here imitation dynamics with *myopic firms*. The imitation mechanism is based on the probability for a firm programmed in a particular strategy to switch toward another strategy. In our case, the probability that firm $p \in N$ switch from strategy *x* towards strategy *y* (i.e., that firm *p* adopts the post idiosyncratic shock strategy of firm *h*) is given by:

$$P_x^y(N) = N_y \emptyset\left[u_y(p) - u_x(p)\right] \text{ if } x \neq y \quad (7)$$

[7] In the rest of the paper, in order to simplify the model, we assume that *f=f(x).*
[8] We assume that $\sum_{y=1} w_{m,p} = 1$, which guarantees that the extent to which the state of each firm depends on the rest of firm is constant.
[9] ρ_{hp} is not required to be stochastic, i.e., the sum of total power exercise does not require be equal to one.

Where $\emptyset$ is a continuously probability distribution function $\emptyset : R \rightarrow [0,1]$[10], $u_y(p)$ correspond to the utility of strategy *y* of firm *p* and $u_x(p)$ is the utility of strategy *x* of firm *p*. Based on assumption 3 we can establish the replication dynamic of strategies *x* and *y* in population *N* given respectively by share population programmed to strategy derivate by time $\dot{N}_x$ and $\dot{N}_y$:

$$\dot{N}_x = \left[\sum_{y\in k} N_y\left(\emptyset\left[u_x(p) - u_y(p)\right] - \emptyset\left[u_y(p) - u_x(p)\right]\right)\right]N_x \tag{8}$$

$$\dot{N}_y = \left[\sum_{y\in k} N_x\left(\emptyset\left[u_y(p) - u_x(p)\right] - \emptyset\left[u_x(p) - u_y(p)\right]\right)\right]N_y \tag{9}$$

Where N_y and N_x are repectively the initial share of population programmed to pure strategy *x* and *y*. At this stage, it is necessary to assume that the replication dynamic of strategy *y* in firm *h* from population *N* cannot be negative meaning that firm *h* will not switch toward strategy *x*.[11] Given these replications dynamic model, we can make the following proposition. We will see in the next section the different necessary condition to this.

Proposition 1 (Propagation effectiveness) *Given the replication dynamic $\dot{N}_x$ and $\dot{N}_y$ of strategies x and y, the x-reviewing firm p switch to the other firm strategy y if and only if:*

$$u_y(p) > u_x(p) \tag{10}$$

4 Propagation

In this section, we describe the general game that we call *idiosyncratic game* arising from the occurrence of aleatory idiosyncratic shock to *hub-firm* and solve it to found condition to the propagation of strategy *y* and so the idiosyncratic shock. We show that the propagation of idiosyncratic shock remains on three non-additive conditions, namely the payoff distribution, the interaction predominance, and the power influence.

4.1 Adaptative strategy as Nash equilibrium

Like we say previously, under the assumption of replicator dynamic, initially firm *h* and firm *p* are initially programmed to strategy *x* (with idiosyncratic differences and different payoffs). Then, an idiosyncratic shock happens on firm *h,* inducing the latter to switch toward

[10] In the rest of the paper, in order to simplify the model, we assume that $\emptyset = \emptyset(x)$.

[11] We assume that *hub-firm* will not switch strategies given the *idiosyncratic game.* A further extension to multipopulational model could lead to relax this assumption.

another strategy from her set of pure strategy. Given these move, firm *p* has now the choice to stay programmed to pure strategy *x*, or to switch towards strategy *y*.

These information's give us the following payoff matrix:

Figure 4: Payoff matrix of *idiosyncratic game*

		Firm h	
		$x(h)$	$y(h)$
Firm p	$x(p)$	$x_x(p)\,;\,x_x(h)$	$x_y(p)\,;\,y_x(h)$
	$y(p)$	$y_x(p)\,;\,x_y(h)$	$y_y(p)\,;\,y_y(h)$

Proposition 1 state that this switch will operate when $u_y(p) > u_x(p)$. So, given the utilities function given by (5) and (6), we can establish the right condition to validate proposition 1 based on the following proposition.

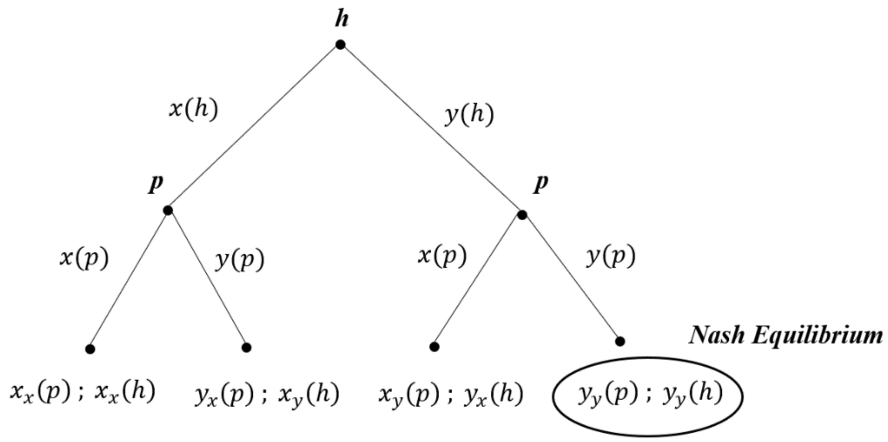
Proposition 2 (Growth of utility function) *For a utility* $u(i)$ *of given firm i with strategy profile s* $\in S$*, and associated payoff* $s(i)$ *:*

$$\frac{\partial u(i)}{\partial s(i)} > 0 \tag{11}$$

For all s(i) *defined on R, meaning that the utility function is strictly increasing on R relatively to s(i).*

The first condition to validate proposition 1 relies on the payoff distributions of strategies *x* and *y*. If the firm *p* payoffs distribution of strategy *y* is better than the payoff distribution of strategy *x*, then $u_y(p) > u_x(p)$. Let imagine an idiosyncratic shock hitting *hub-firm. T*hen $y_y(h) > y_x(h)$ because *h* will adapt his strategy in order to gain more given the shock. This mean given proposition 2, that proposition 1 could be valid if $y_y(p) > x_y(p)$ meaning that given that firm *h* plays *y*, firm *p* has interest in playing strategy *y*. This situation corresponds to a Nash equilibrium (see figure 5) Since in the descried game firm *h* play first, we can describe the game as follow:

Figure 5: Extensive form of *idiosyncratic game* with Nash equilibrium

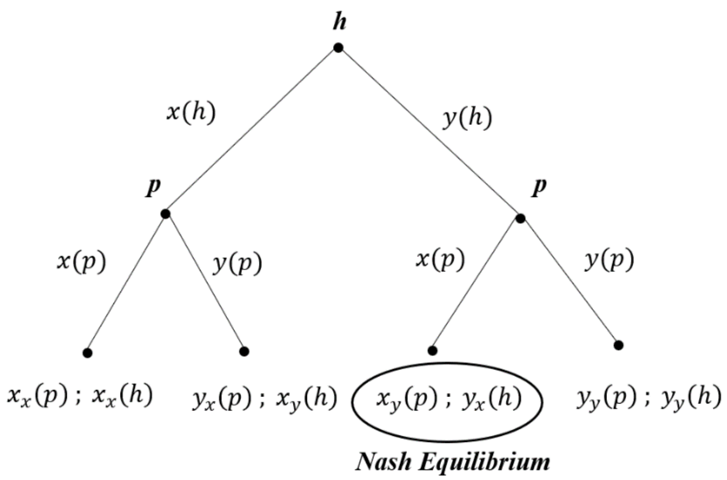


This mean that when the adaptative strategy to an idiosyncratic shock correspond to a Nash equilibrium, the idiosyncratic shock propagates. We can observe this case when idiosyncratic shock is positive because firms are part of a network and synergy give them more gain.

4.2 Invalid Nash equilibrium

Now, it is possible that the adaptative strategy to an idiosyncratic shock do not correspond to a Nash equilibrium because $x_y(p) > y_y(p)$. This mean that the gains of firm *p* to switch towards strategy *y* are less good than the ones associated to strategy *x*. This case corresponds to the following game:

Figure 6: Extensive form of *idiosyncratic game* without Nash equilibrium



Then, we cannot explain the propagation of idiosyncratic shock by the payoff distribution of different strategies. In that case, the propagation of idiosyncratic shock relies on the second aspect that is the presence of interaction materialized by the constant w_{hp}.

For a given level of interaction w_{hp}, the firm *p* will integrate the gain of different strategies on firm *h*, if w_{hp} is sufficiently high, then the utility of firm *p* for a strategy giving her a less good payoff than another one can be superior as $u_y(p) > u_x(p)$. Following Lemma 1 available in appendix C, idiosyncratic shock will propagate for all:

$$w_{hp} > \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h)} \tag{12}$$

This mean that the interaction between firm plays a role in moving the Nash equilibrium towards adaptative strategies to an idiosyncratic shock.

4.3 Insufficient interaction

In the case where the adaptative strategies to an idiosyncratic shock do not correspond to a Nash equilibrium, it is possible firm's interaction may not be sufficient to permit the propagation of adaptative strategies and so idiosyncratic shocks. In fact, two scenarios are possibles. Either the payoff distribution of strategies does not allow interaction to compensate them, so there is no level of interaction that can allow the propagation of adaptative strategies, or the effective interaction constant w_{hp} is not sufficient to compensate the payoffs distribution of strategies meaning that $w_{hp} < \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h)}$.

Given the assumption of replicator and following the general framework, we always have $y_y(h) > y_x(h)$. This mean that it's in the interest of the *hub-firm* that the *partner firm* follow the same strategy (with idiosyncratic differences and different payoff) because the coordination in the network will give a better payoff to firm *h*. So, if interaction is no longer sufficient to propagate the adaptative strategy. The firm *h* exercise power in order to propagate the strategy *y* trough ρ_{hp}. In our model this is translate into a coefficient β_1=0 and $\beta_2 = 1$. Following Lemma 2 available in appendix D strategy *y* will propagate if:

$$\rho_{hp} > \frac{\ln\left(x_y(p)^3 - y_y(p)^3 + w_{hp}\left(x_y(p).y_x(h) - y_y(p).y_y(h)\right)\right)}{k} \tag{13}$$

Power will replace the interaction relation and compensate the initial payoff distribution of different strategies. If there is now level of power permitting the propagation of idiosyncratic shock, then shock do not propagate.

If there exist a level of interaction $w_{hp} > \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h))}$, for which the strategy *y* could propagate but that the effective level of interaction is lower than that, then we have the following theorem.

Theorem 1 (Interaction and power relation) *if the effective interaction coefficient* w_{hp} *is inferior to the theoretical coefficient* w_{hp} *permitting the propagation of the adaptative strategy y, then, the lower* w_{hp} *is, the higher* ρ_{hp} *need to be to permit the propagation of the adaptative strategy and vice versa.*

5 Simulations

In this section, we illustrate the different case sketches in section 4 in order to appreciate the evolution of strategy *y* in the population. We start from our initial *idiosyncratic game* depict in section 4 and analyze how positive and negative idiosyncratic shock could propagate.

5.1 Positive idiosyncratic shock

The first case we simulate is the one of a positive idiosyncratic shock. Following the replicator dynamic assumption, *hub-firm* and *partner firm* are both programmed in strategy *x* (with idiosyncratic differences and different payoffs), *hub-firm* initially gain a payoff of 10 using strategy *x* and *partner firm* gain a payoff of 7. Once the idiosyncratic shock occurred, the payoff distribution change as follow:

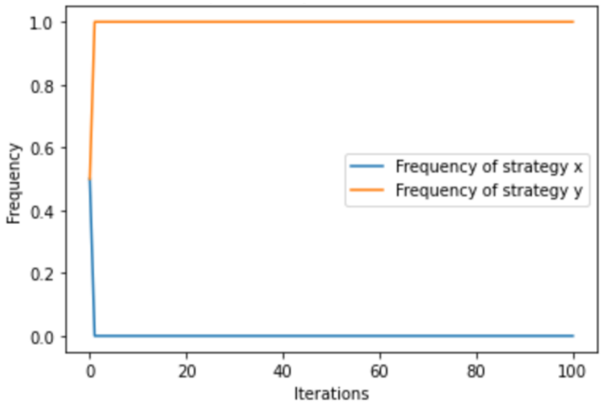
Figure 7: Payoff matrix of *idiosyncratic game* with Nash equilibrium

		Firm h	
		$x(h)$	$y(h)$
Firm p	$x(p)$	8;12	10; 13
	$y(p)$	9;11	12;15

In this case, the Nash equilibrium is situated in (12;15) corresponding to adaptative strategies. Following what we said in section 4, when the Nash equilibrium is located on adaptative strategies, $u_y(p) > u_x(p)$ validating proposition 1. So, Strategy *y* and so the idiosyncratic shock should propagate. In order to illustrate that, we fix $w_{hp} = 0$ and $\beta_1 = \beta_2 = 0$, proving that the propagation only rely on the Nash equilibrium.

Since there are two strategies (*x* and *y*), firm h programmed in strategy *y* and firm *p* programmed in strategy *x*, the initial population state of both strategy Ny and Nx are 0.5. iterations repeat itself 100 times. To isolate the effect of choices probabilities, we assume that all review rates are constantly equal to one. We obtain the following evolution of strategies *x* and *y*.

Figure 8: Simulation if the propagation of idiosyncratic shock relying on Nash equilibrium



As predicted in section 4, the *partner firm* follows the Nash equilibrium, and based on the payoff distribution of strategies, *x* and *y* decide to switch from strategy *x* towards strategy *y*.

5.2 Negative idiosyncratic shock

The second case we simulate is the one of a negative idiosyncratic shock. Following the replicator dynamic assumption, *hub-firm* and *partner firm* are both programmed in strategy *x* (with idiosyncratic differences and different payoffs), *hub-firm* initially gain a payoff of 10 using strategy *x* and *partner firm* gain a payoff of 7. Once the idiosyncratic shock occurred, the payoff distribution change as follow:

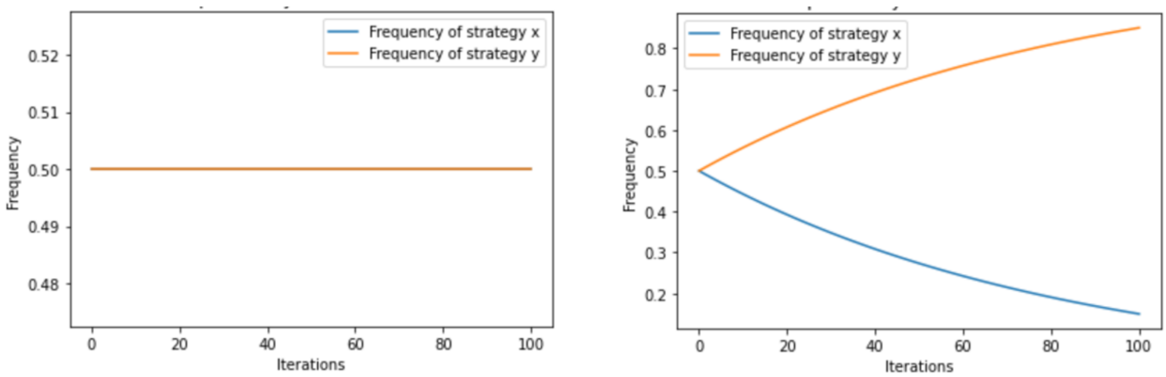
Figure 9: Payoff matrix of *idiosyncratic game* without Nash equilibrium

		Firm h	
		$x(h)$	$y(h)$
Firm p	$x(p)$	4;5	3; 8
	$y(p)$	6;6	2;25

In this case there is now Nash equilibrium. So, in order to validate the proposition 1 and following the Lemma 1, the interaction coefficient w_{hp} need to be superior to 0.730. we made two simulations. One where we fix $w_{hp} = 0.729$ and an another where $w_{hp} = 0.731$. We keep $\beta_1 = \beta_2 = 0$, proving that the propagation only relies on the interaction coefficient.

As in the precedent example, the initial population state of both strategy Ny and Nx are 0.5. iterations repeat itself 100 times.

Figure 10: Simulation of the propagation of idiosyncratic shock relying on interaction



As we can see, in the first case where $w_{hp} = 0.729$ i.e., lower than the theoretical needed interaction coefficient, the population share programmed in strategy *y* stay at 0.5 corresponding to the firm *h*. This mean that the strategy *y* does not propagate.

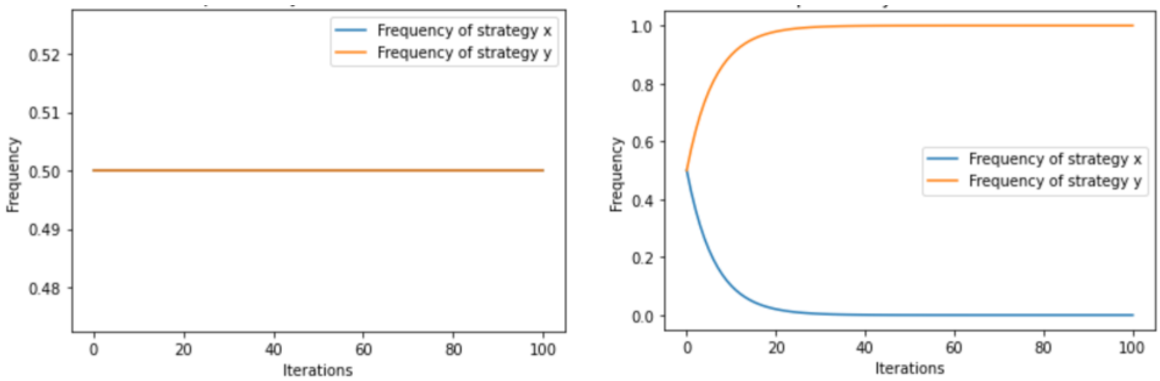
On the other hand, when $w_{hp} = 0.731$, i.e. higher than the theoretical needed interaction coefficient, the population share programmed in strategy *y* increase until it reaches the totality of the population. These results illustrate the fact that when the payoff distribution of adaptative strategies do not constitute a Nash equilibrium, interaction coefficient take the lead and if sufficiently high result in propagation of strategy *y*.

We can now move on the situation where the interaction coefficient is no sufficient to provoked the propagation. Following the precedent example, if we set $w_{hp} = 0.2$ meaning a level too low to influence the propagation, then and based on Lemma 2, the power coefficient ρ_{hp} need to be superior to 0.525.

As previously, we made two simulations. One where we fix $\rho_{hp} = 0.524$ and an another where $\rho_{hp} = 0.526$. This time, firm h exercises is power meaning that $\beta_1 = 0$ and $\beta_2 = 1$ proving that the propagation only relies on the power coefficient.

As in the precedent example, the initial population state of both strategy N_y and N_x are 0.5. iterations repeat itself 100 times. We still assume that all review rates are constantly equal to one.

Figure 11: Simulation of the propagation of idiosyncratic shock relying on power



As we can see, in the first case where $\rho_{hp} = 0.524$, i.e. lower than the theoretical needed power coefficient, the population share programmed in strategy *y* stay at 0.5 corresponding to the firm *h*. This mean that the strategy *y* does not propagate.

As in the previous case, On the other hand, when $\rho_{hp} = 0.526$, i.e. higher than the theoretical needed power coefficient the population share programmed in strategy *y* increase until it reaches the totality of the population. These results illustrate the fact that when the payoff distribution of adaptative strategies do not constitute a Nash equilibrium, interaction coefficient take the lead and if sufficiently high result in propagation of strategy *y*.

6 Conclusion

This paper presented a firm network framework augmented by replicator dynamic assumption nesting a variety of *idiosyncratic game*. Under the assumption that idiosyncratic shock leads to a change of strategy of firm to the one who suffers the shock (and that the shock is suffer by what we call a *hub-firm*), our main results provide a fairly characterization of the propagation of idiosyncratic shock. These highlight that the importance of network structure in the propagation of idiosyncratic shock and the fact that this propagation relies on three non-additive conditions. Our characterization underlines that idiosyncratic shock propagate when the payoff distribution of adaptative strategy constitutes a Nash equilibrium for the firms, when the interaction between firm is sufficiently high, or when the degree power exercises by the *hub-firm* is at a sufficient level. These results show that the propagation of idiosyncratic shock can be resume as *idiosyncratic game* characterized by a perfect information cooperation game, that can be oriented by the *hub-firm* under certain condition, and almost always resulting in the least bad possible way for the *hub-firm.*

These work opens fields to a variety of future research. Firstly, the reduced framework can be extended to a multipopulational version, complexifying the intra-network relation and giving more relief to the study. Secondly, for our part, we focus on an idiosyncratic shock hitting the *hub-firm*, take an interest on shock located on *partner firm* could be useful to understand fully the propagation of idiosyncratic shock. Thirdly, our framework consists on a network composed by inter-firm relation, these relations rely on interaction, and power. Further researcher in completing the nature of these relation and on other relation could be interesting. Finally, the work opens the fields to empirical work on the intra-network relation and on the replication dynamic of strategies within network.

A Proof of proposition 1

We start from the replication dynamic equation of strategy *y*.

$$\dot{N}_y = \left[\sum_{y \in k} N_x\left(\emptyset\left[u_y(p) - u_x(p)\right] - \emptyset\left[u_x(p) - u_y(p)\right]\right)\right] N_y$$

Following this replication dynamic equation, if the firms switch from strategy *x* to strategy *y*, then $\dot{N}_y > 0$. Let fix $\emptyset = \emptyset(x)$ a continuously probability distribution function.

$$\dot{N}_y = \left[\sum_{y \in k} N_x\left(\emptyset\left[u_y(p) - u_x(p)\right] - \emptyset\left[u_x(p) - u_y(p)\right]\right)\right] N_y > 0$$

Since $N_x > 0$ and $N_y > 0$, $\dot{N}_y > 0$ if:

$$\emptyset\left[u_y(p) - u_x(p)\right] - \emptyset\left[u_x(p) - u_y(p)\right] > 0$$

$$<=> \ \left[u_y(p) - u_x(p)\right] - \left[u_x(p) - u_y(p)\right] > 0$$

$$<=> \ u_y(p) > u_x(p)$$

B Proof of proposition 2

The general utility function of firm *i* programmed in strategy *s(i)* can be describe as follow:

$$u(i) = s(i)^3 + s(i).f\left(\sum_{y=1} w_{hp}.t(j)\right) + \beta_1.e^{\left(\sum_{y=1} k\rho_{hp}\right)}$$

Where $t(j)$ is the strategy of player *j*.
The utility function is strictly increasing with respect to *s(i)* if his derivative with respect to *s(i)* is superior to zero.

$$\frac{\partial u(i)}{\partial s(i)} = 3s(i)^2 + f\left(\sum_{y=1} w_{hp}.t(j)\right)$$

Let *f*=*f(x)* and since $\sum_{y=1} w_{hp}.t(j)$ is constant:

$$\frac{\partial u(i)}{\partial s(i)} > 0$$

C Lemma 1

The following lemma is used in section 4.

Lemma 1 *if payoff distribution of adaptative strategy is not a Nash equilibrium, then firm p switch from strategy x toward strategy y if:*

$$w_{hp} > \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h)}$$

Proof of Lemma 1. Firm *p* switch from strategy *x* to *y* if:

$$u_y(p) > u_x(p)$$

If payoff distribution of adaptative strategy is not a Nash equilibrium, then $u_y(p) > u_x(p)$ if

$$y_y(p)^3 + y_y(p).f\left(\sum_{y=1} w_{hp}.y_y(h)\right) + \beta_2.e^{\left(\sum_{y=1} k\rho_{hp}\right)}$$

$$> x_y(p)^3 + x_y(p).f\left(\sum_{y=1} w_{hp}.y_x(h)\right) + \beta_1.e^{\left(\sum_{y=1} k\rho_{hp}\right)}$$

$$<=> y_y(p)^3 + y_y(p).f\left(\sum_{y=1} w_{hp}.y_y(h)\right) > x_y(p)^3 + x_y(p).f\left(\sum_{y=1} w_{hp}.y_x(h)\right)$$

$$<=> w_{hp} > \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h)}$$

D Lemma 2

The following Lemma is used in section 4.

Lemma 2 *if payoff distribution of adaptative strategy is not a Nash equilibrium and that* $w_{hp} < \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h)}$, *then firm p switch from strategy x toward strategy y if:*

$$\rho_{hp} > \frac{\ln\left(x_y(p)^3 - y_y(p)^3 + w_{hp}\left(x_y(p).y_x(h) - y_y(p).y_y(h)\right)\right)}{k}$$

Proof of Lemma 2 Firm *p* switch from strategy *x* to *y* if:

$$u_y(p) > u_x(p)$$

If payoff distribution of adaptative strategy is not a Nash equilibrium and $w_{hp} < \frac{x_y(p)^3 - y_y(p)^3}{(y_y(p).y_y(h) - x_y(p).y_x(h)}$ then $u_y(p) > u_x(p)$ if

$$y_y(p)^3 + y_y(p).f\left(\sum_{y=1} w_{hp}.y_y(h)\right) + \beta_2.e^{\left(\sum_{y=1} k\rho_{hp}\right)}$$
$$> x_y(p)^3 + x_y(p).f\left(\sum_{y=1} w_{hp}.y_x(h)\right) + \beta_1.e^{\left(\sum_{y=1} k\rho_{hp}\right)}$$

$$<=> \beta_2.e^{\left(\sum_{y=1} k\rho_{hp}\right)} - \beta_1.e^{\left(\sum_{y=1} k\rho_{hp}\right)}$$
$$> x_y(p)^3 - y_y(p)^3 + w_{hp}\left(x_y(p).y_x(h) - y_y(p).y_y(h)\right)$$

$$<=> \rho_{hp} > \frac{\ln\left(x_y(p)^3 - y_y(p)^3 + w_{hp}\left(x_y(p).y_x(h) - y_y(p).y_y(h)\right)\right)}{k}$$

Since β_1 *=0* and $\beta_2 = 1$

E Proof of theorem 1

The power coefficient is given by:

$$\rho_{hp} > \frac{\ln\left(x_y(p)^3 - y_y(p)^3 + w_{hp}\left(x_y(p).y_x(h) - y_y(p).y_y(h)\right)\right)}{k}$$

In *idiosyncratic game* where the power coefficient needs to be activated, we have:

$$y_y(h) > y_x(h) \geq x_y(p) > y_y(p)$$

In a way that:

$$x_y(p).y_x(h) - y_y(p).y_y(h) < 0$$

Given that, when w_{hp} increase, the term $w_{hp}\left(x_y(p).y_x(h) - y_y(p).y_y(h)\right)$ decrease and so the coefficient ρ_{hp}.

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